Fuzzy conjunction (triangular norm, t-norm)

binary operation $\underset{\cdot}{\wedge} : [0,1]^2 \to [0,1]$ such that, for all $\alpha, \beta, \gamma \in [0,1]$:

$\alpha \underset{\cdot}{\wedge} \beta = \beta \underset{\cdot}{\wedge} \alpha$	(commutativity)	(T1)
$\alpha \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\wedge} \gamma) = (\alpha \underset{\cdot}{\wedge} \beta) \underset{\cdot}{\wedge} \gamma$	(associativity)	(T2)
$\beta \leq \gamma \Rightarrow \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} \gamma$	(monotonicity)	(T3)
$\alpha \underset{\cdot}{\wedge} 1 = \alpha$	(boundary condition)	(T4)

Theorem: $\alpha \underset{\cdot}{\wedge} 0 = 0$.

Proof: Using (T3) and (T4): $\alpha \underset{\cdot}{\wedge} 0 \overset{(\mathrm{T3})}{\leq} 1 \underset{\cdot}{\wedge} 0 \overset{(\mathrm{T4})}{=} 0$.

Examples of fuzzy conjunctions

- **Standard** conjunction (**min**, **Gödel**, **Zadeh**, . . .):

$$\alpha \underset{S}{\wedge} \beta = \min(\alpha, \beta).$$

- **Łukasiewicz** conjunction (**Giles**, **bold**, . . .):

$$\alpha \underset{L}{\wedge} \beta = \begin{cases} \alpha + \beta - 1 & \text{if } \alpha + \beta - 1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

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$$\alpha \underset{P}{\wedge} \beta = \alpha \cdot \beta.$$

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$$\alpha \underset{D}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1, \\ \beta & \text{if } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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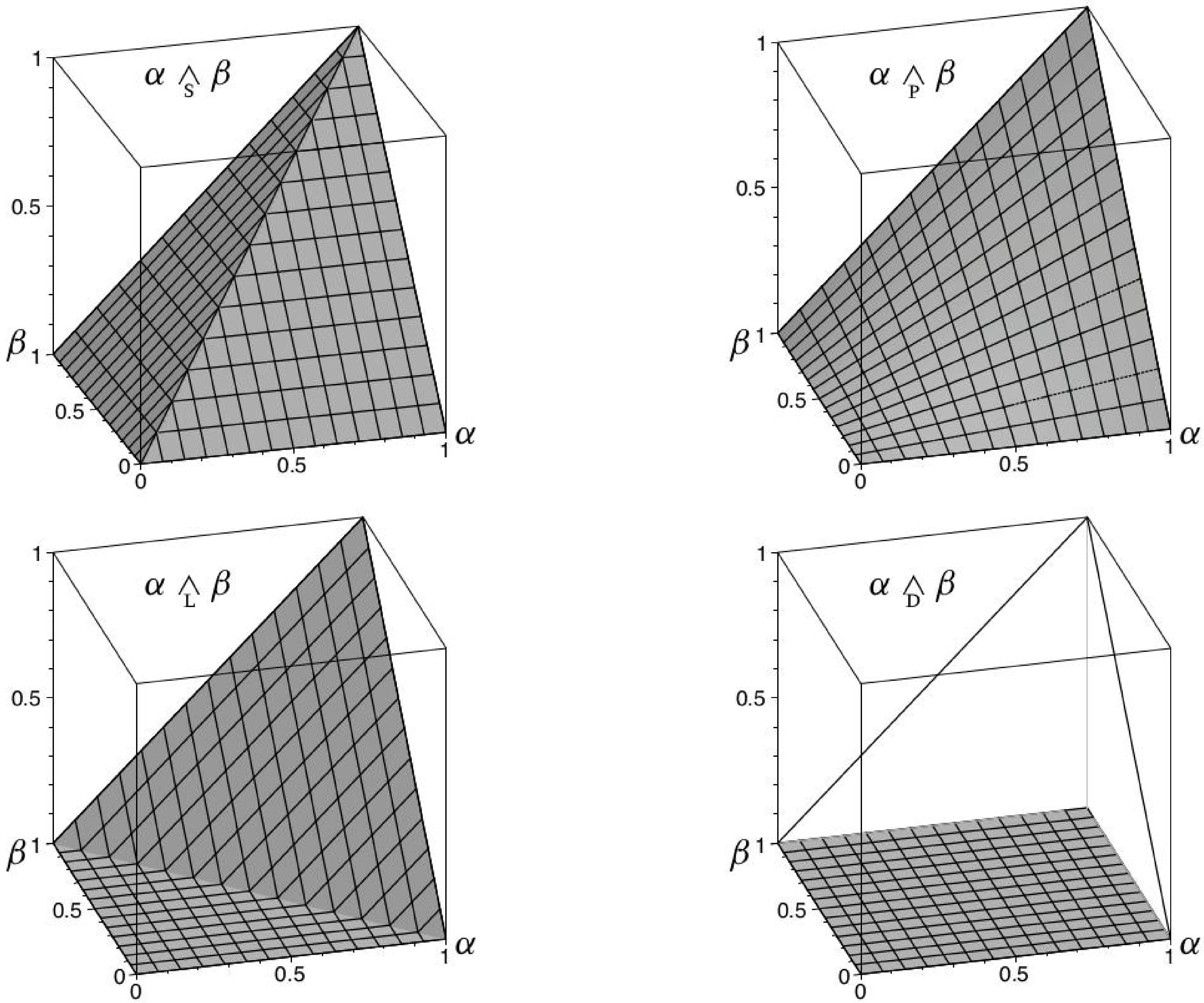

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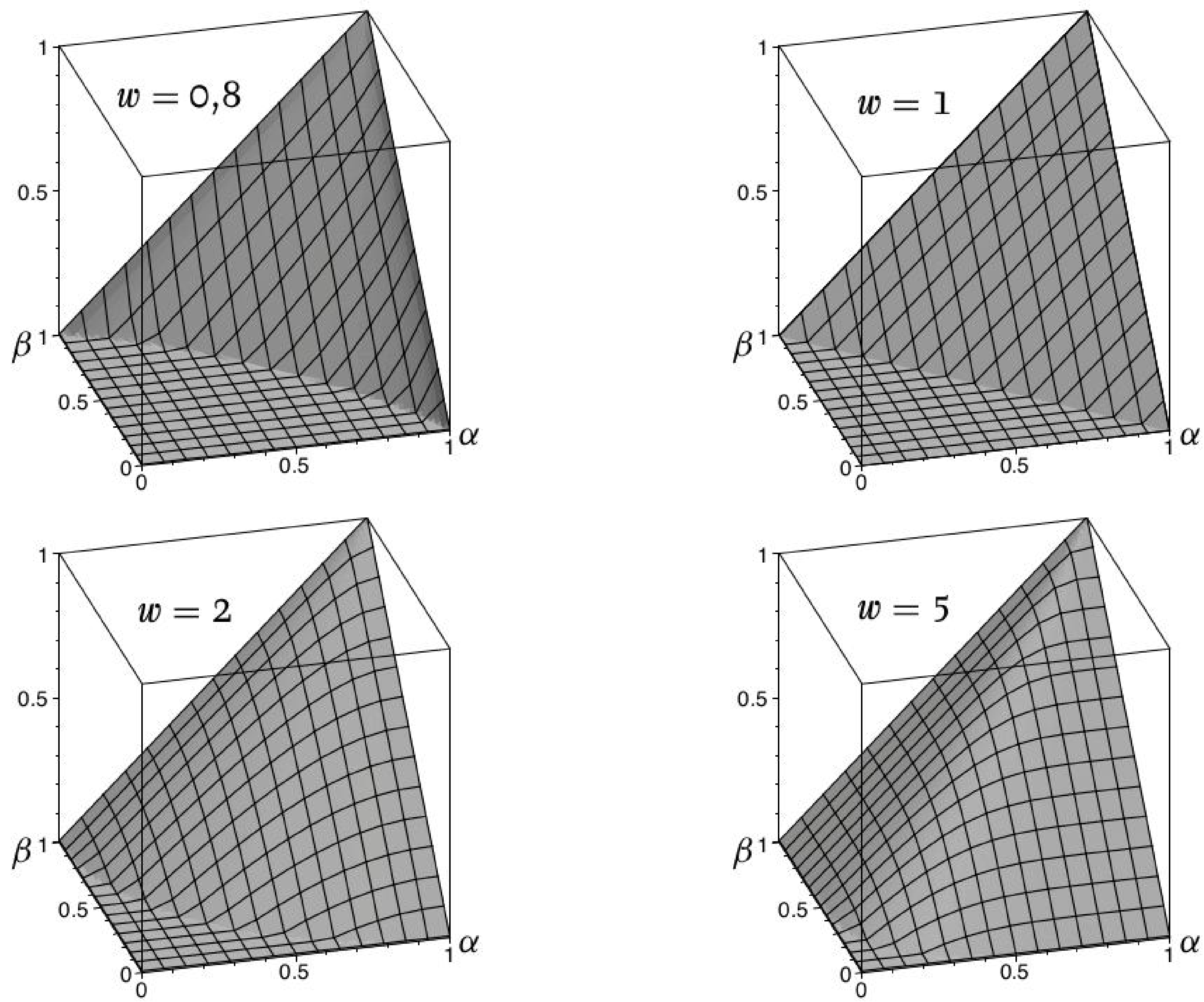

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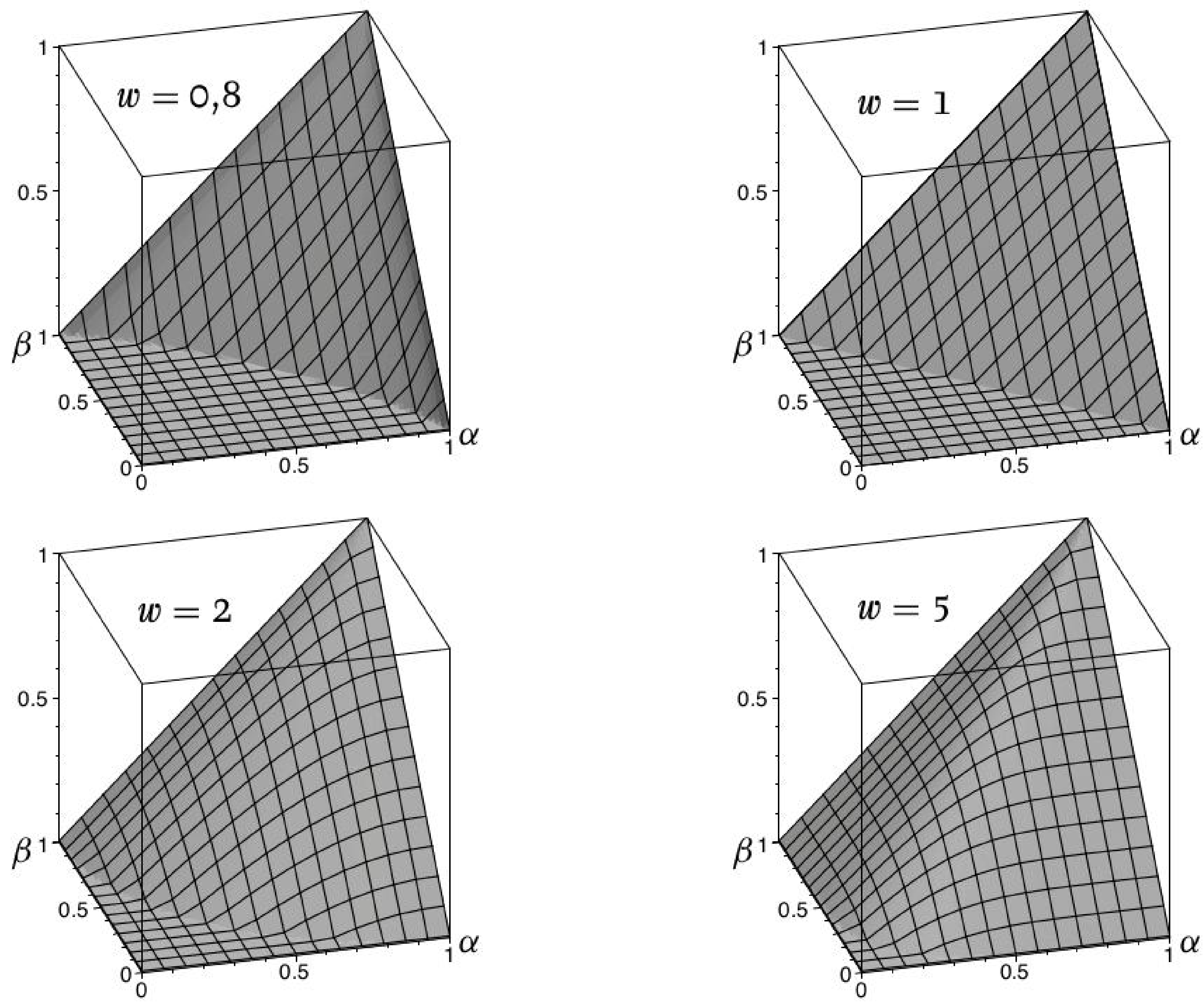

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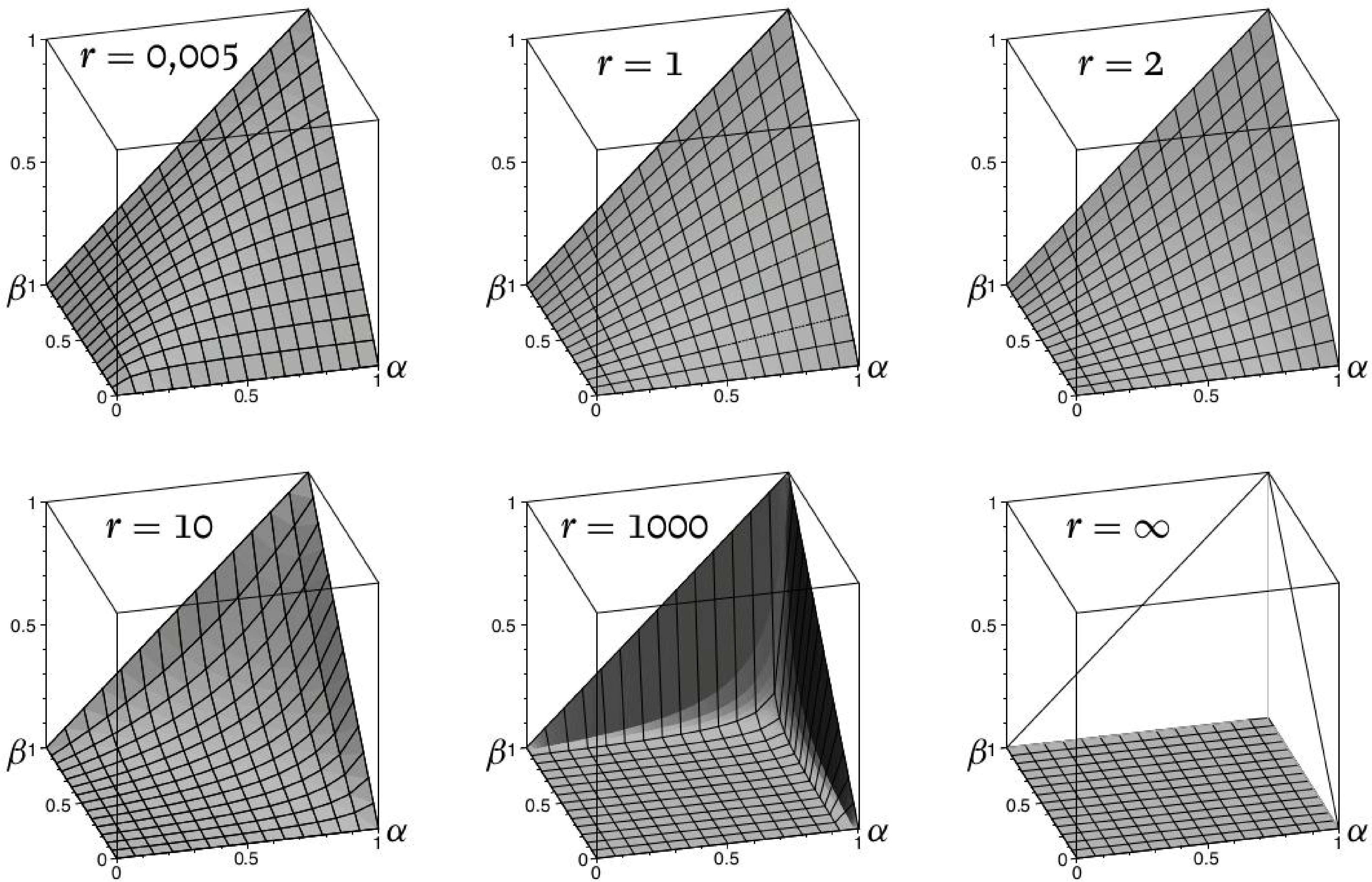

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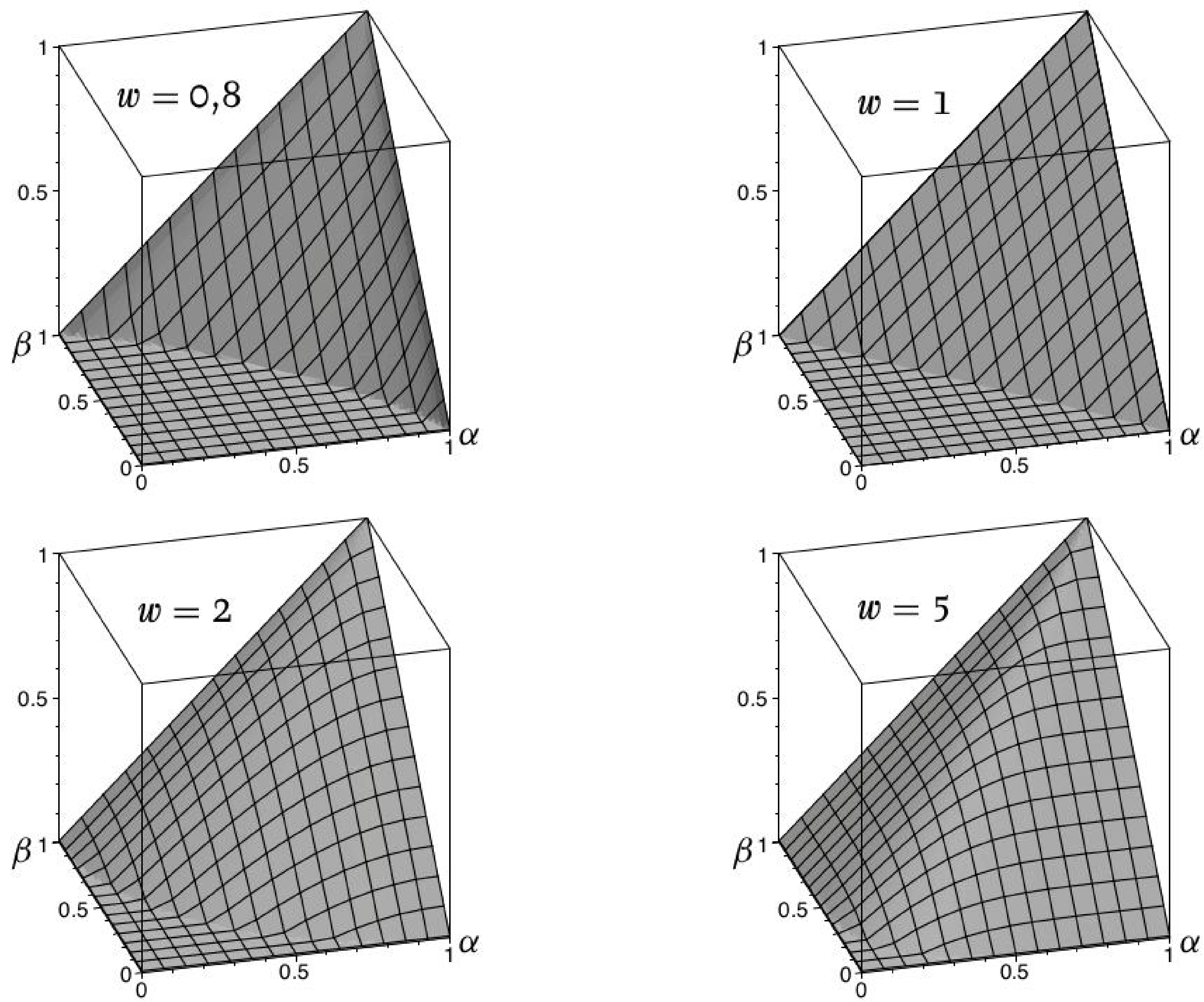

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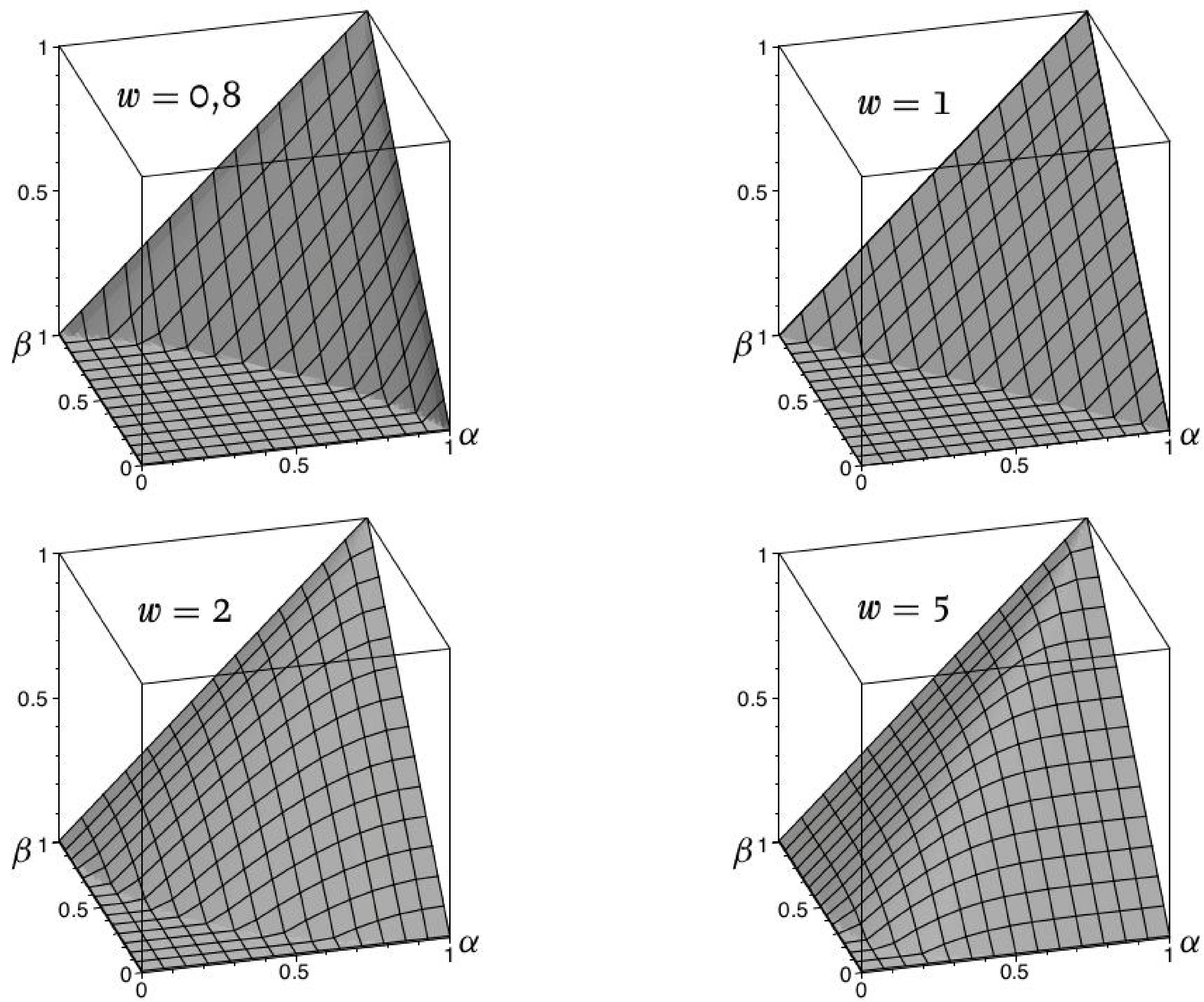

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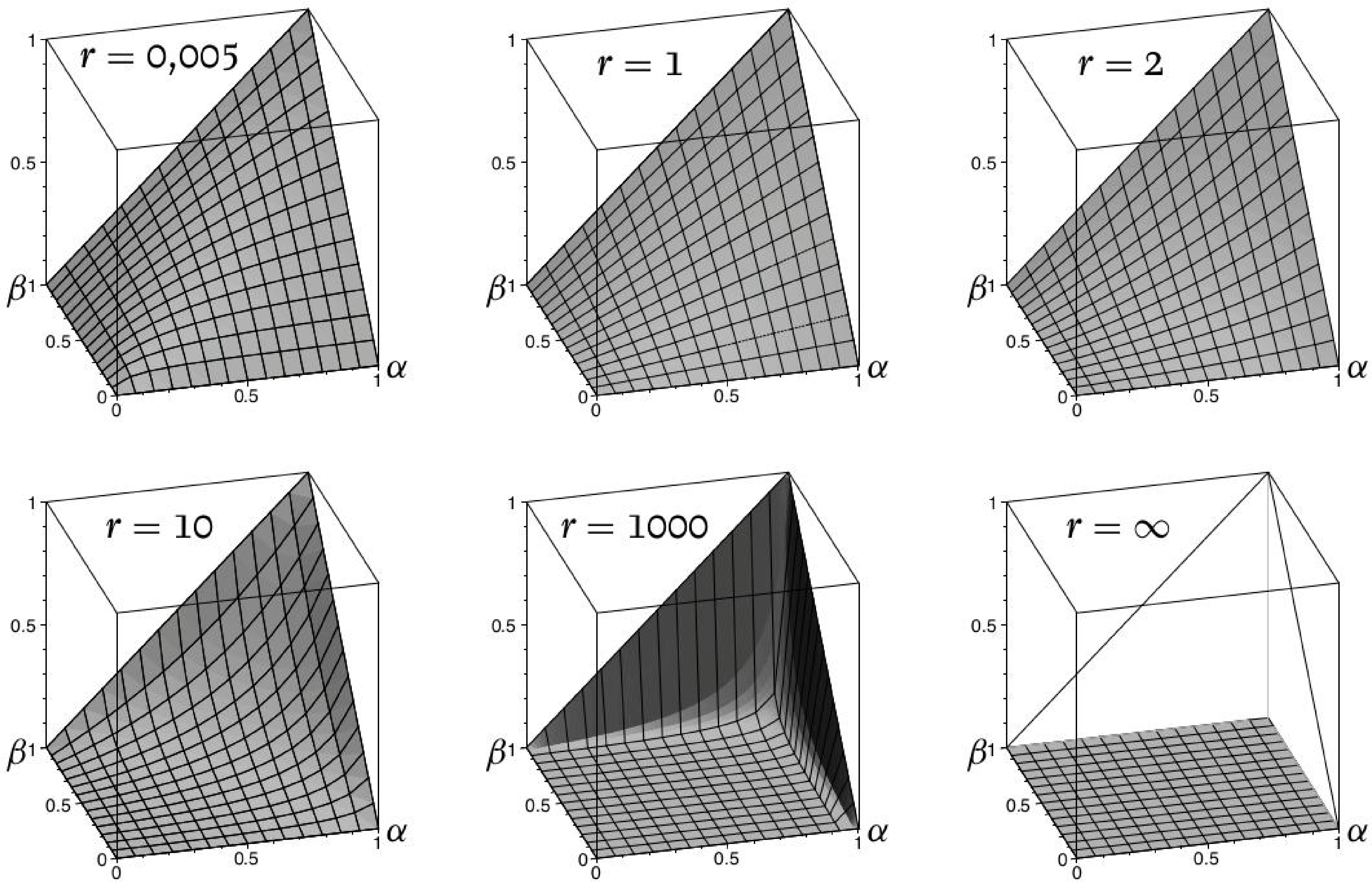

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$$\forall \alpha, \beta \in [0,1]: \quad \alpha \underset{\mathrm{D}}{\wedge} \beta \le \alpha \underset{\cdot}{\wedge} \beta \le \alpha \underset{\mathrm{S}}{\wedge} \beta.$$

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Theorem: Let $\underset{1}{\wedge}$ be a fuzzy conjunction and $i : [0,1] \to [0,1]$ be an increasing bijection. Then the operation $\underset{2}{\wedge} : [0,1]^2 \to [0,1]$ defined by

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Continuous fuzzy conjunction $\underset{\cdot}{\wedge}$ is

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- **strict** if
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- **nilpotent** if it is Archimedean and not strict.

Example: Product conjunction is strict, Łukasiewicz conjunction is nilpotent, standard and drastic conjunctions are not Archimedean (the standard one violates (TA), the drastic one is not continuous).

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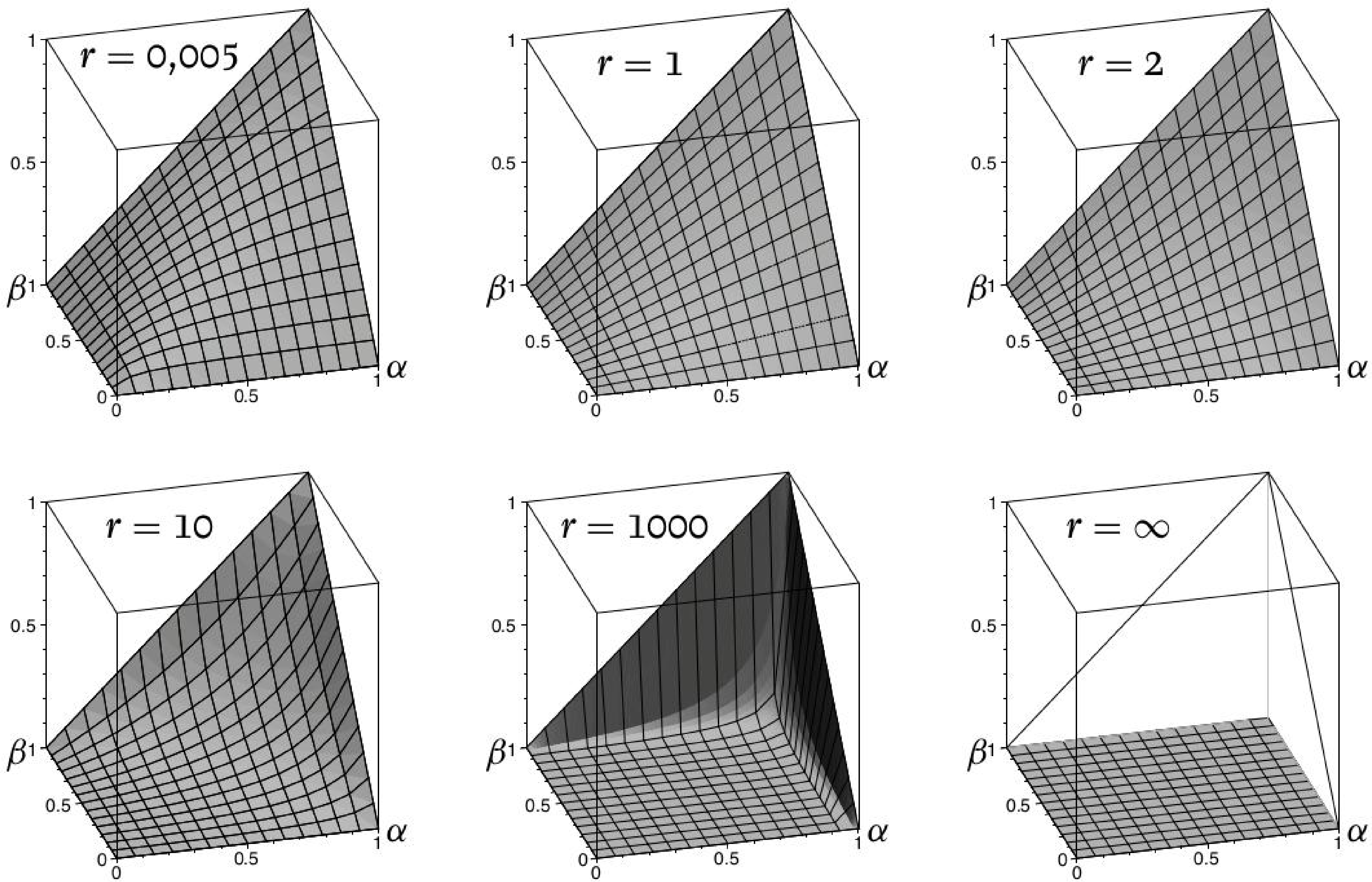

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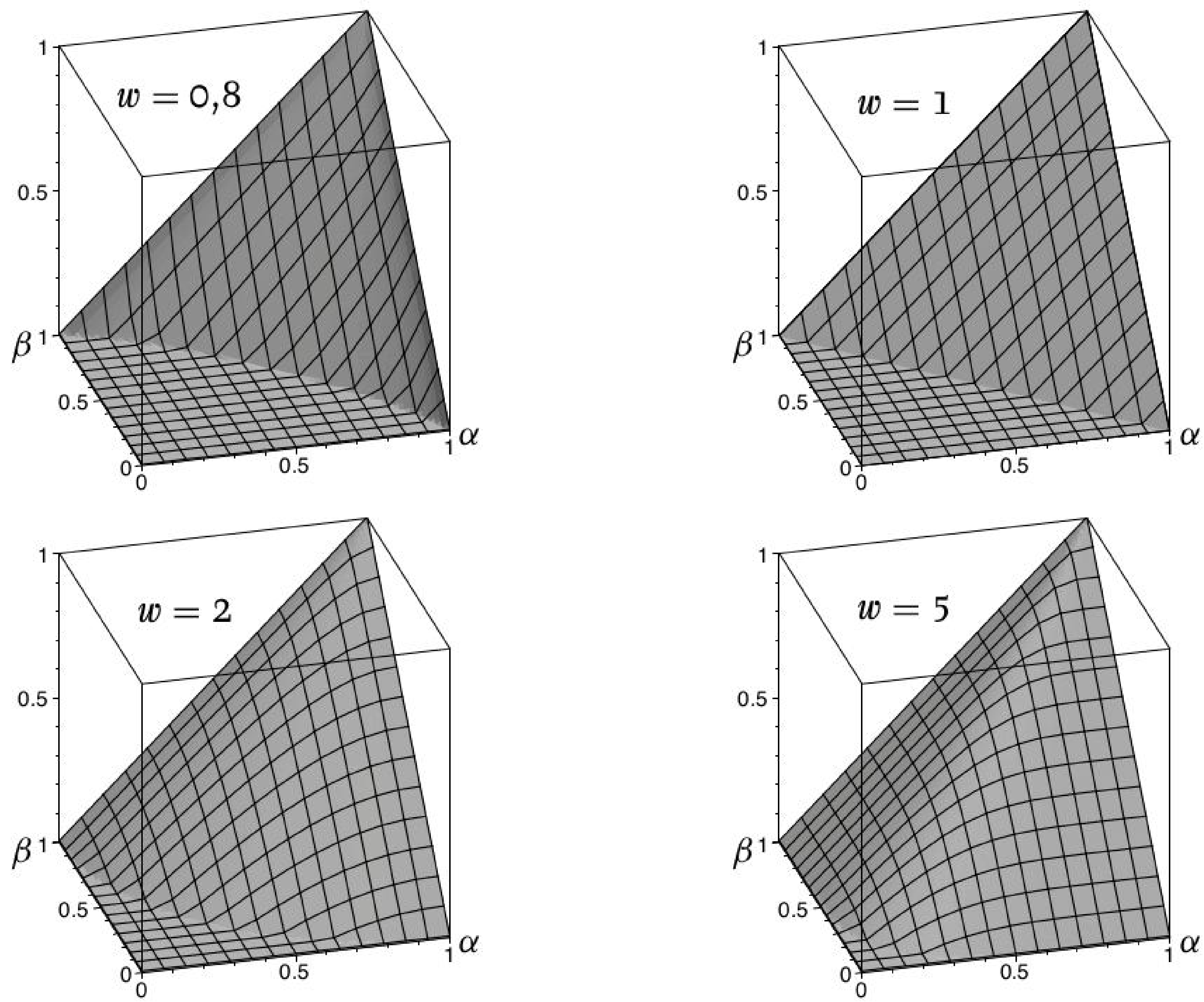

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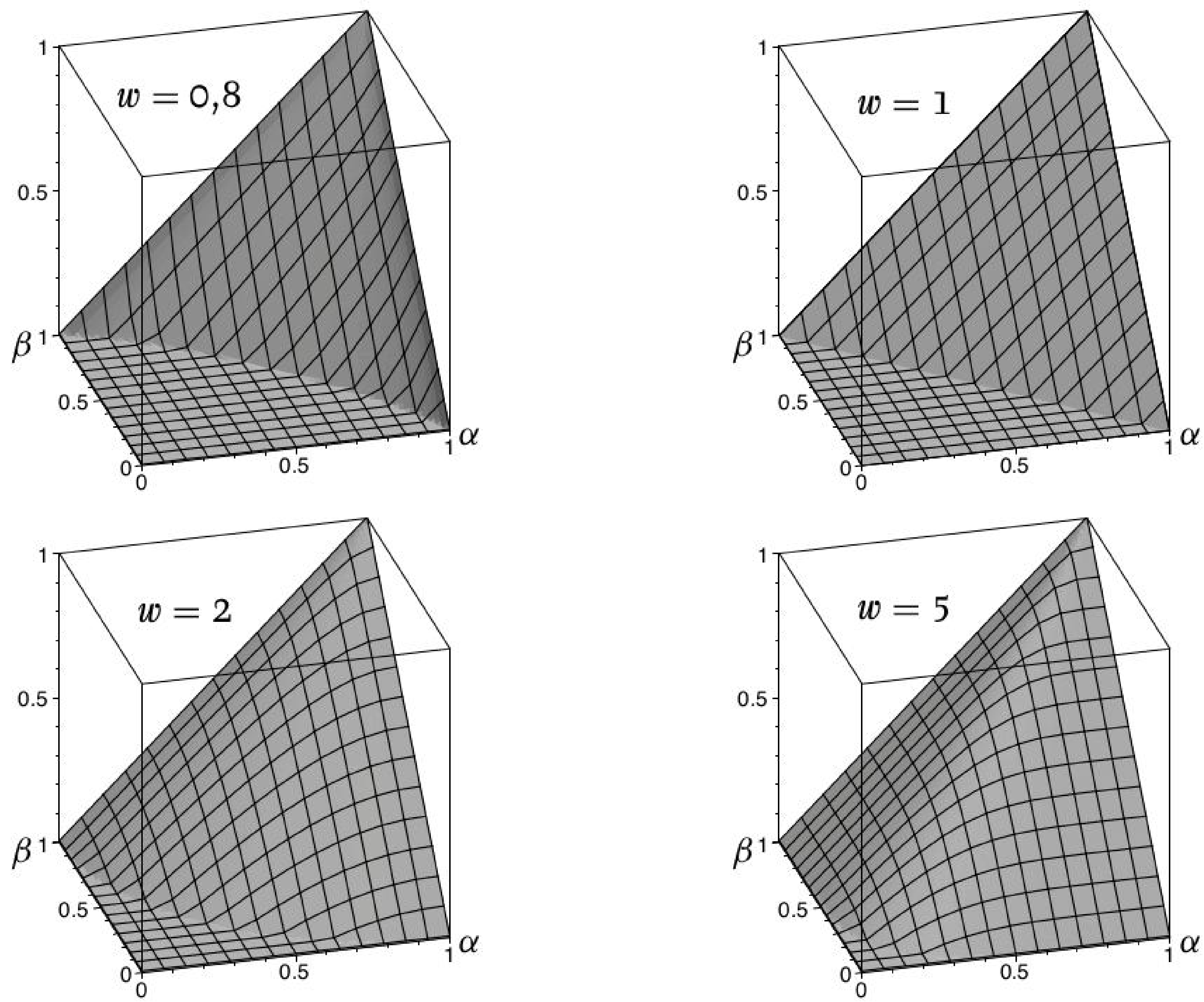

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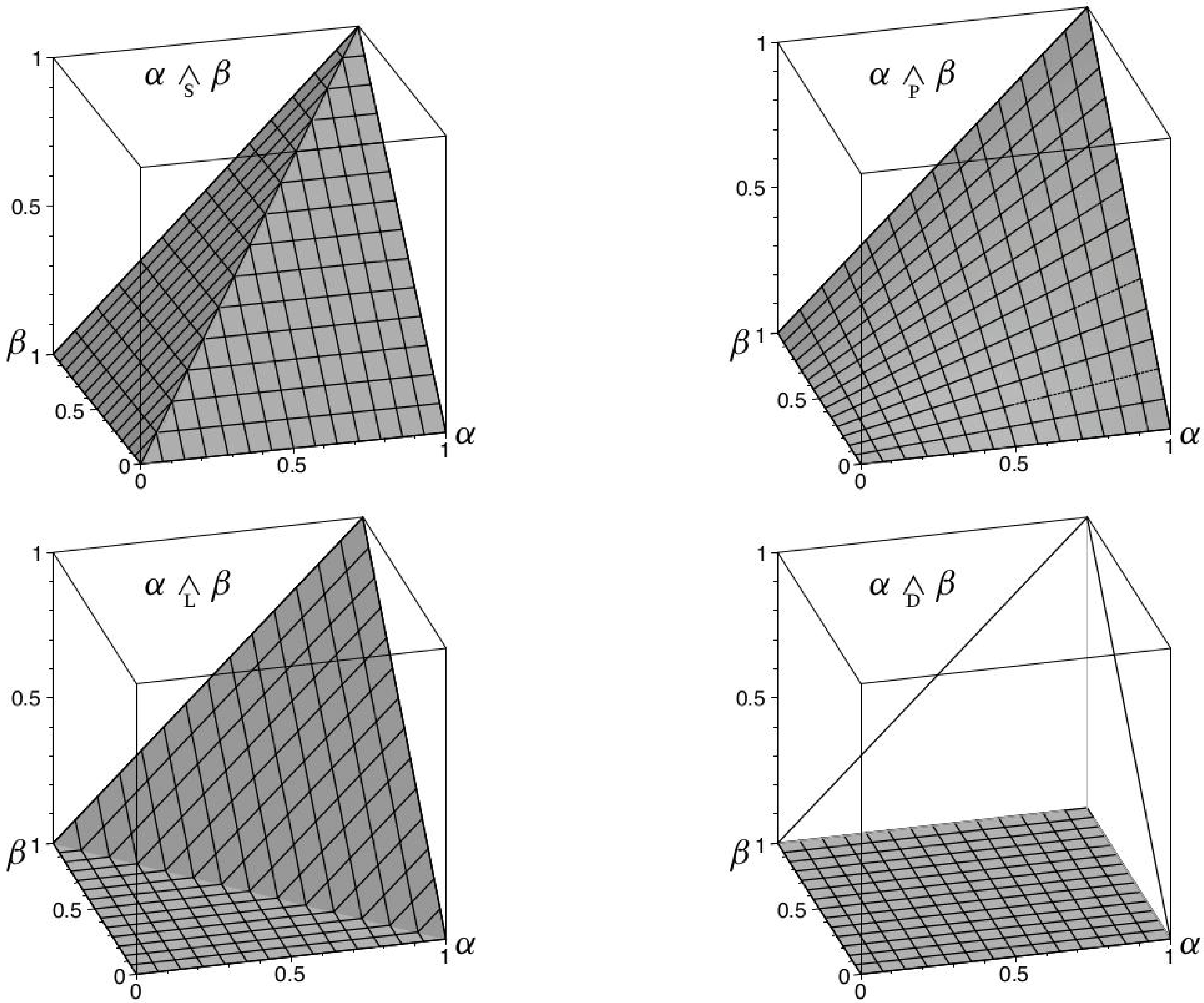

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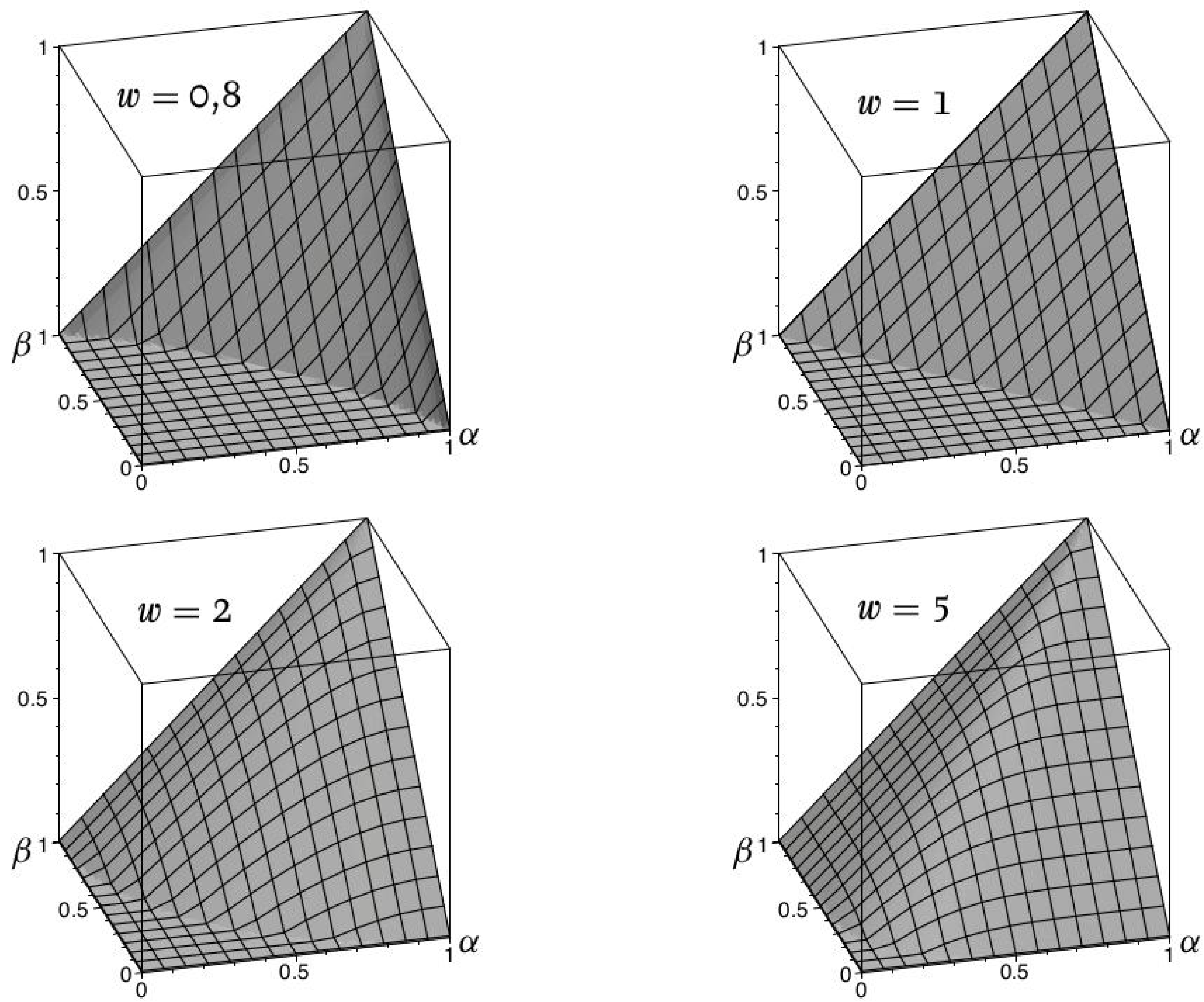

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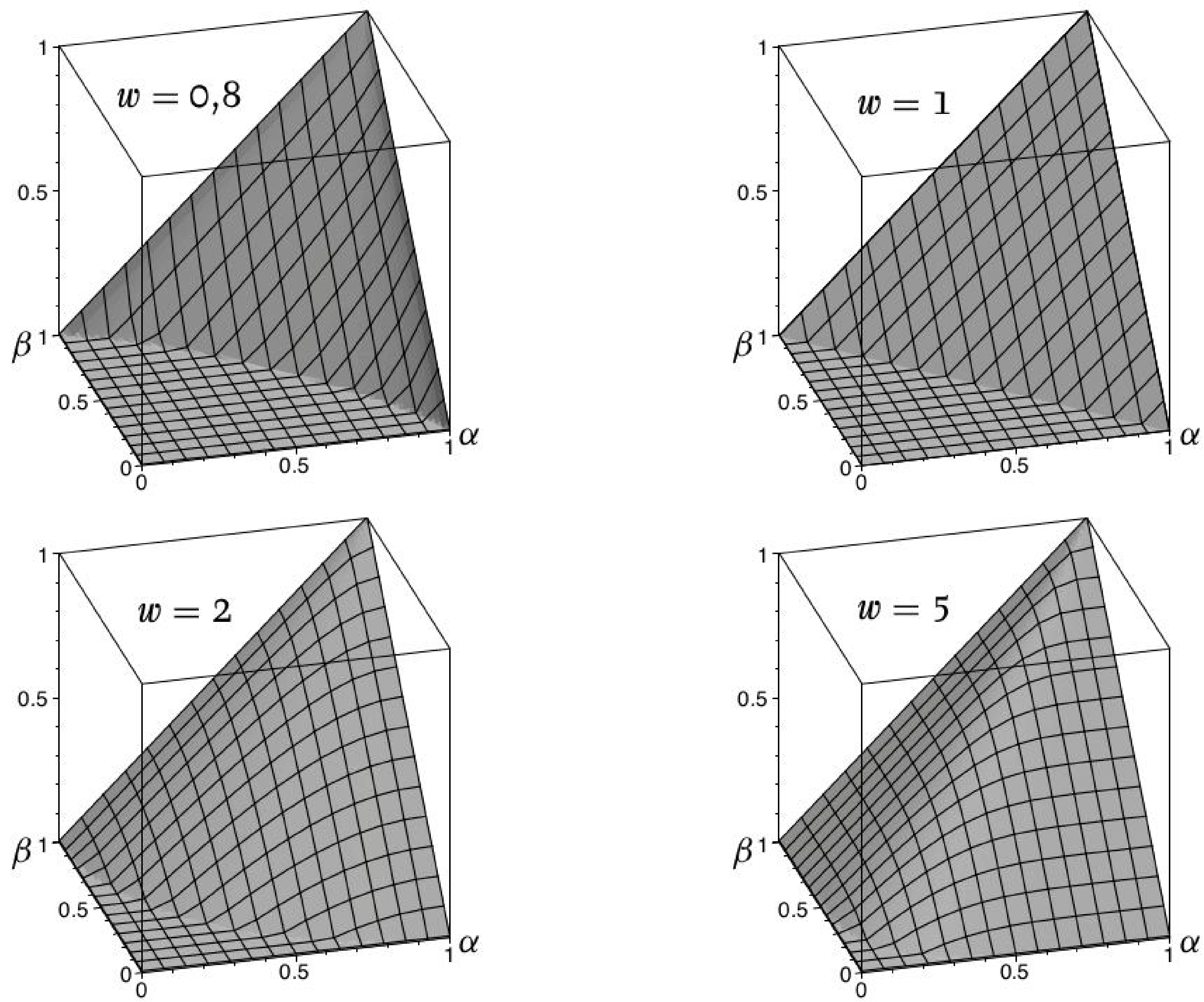

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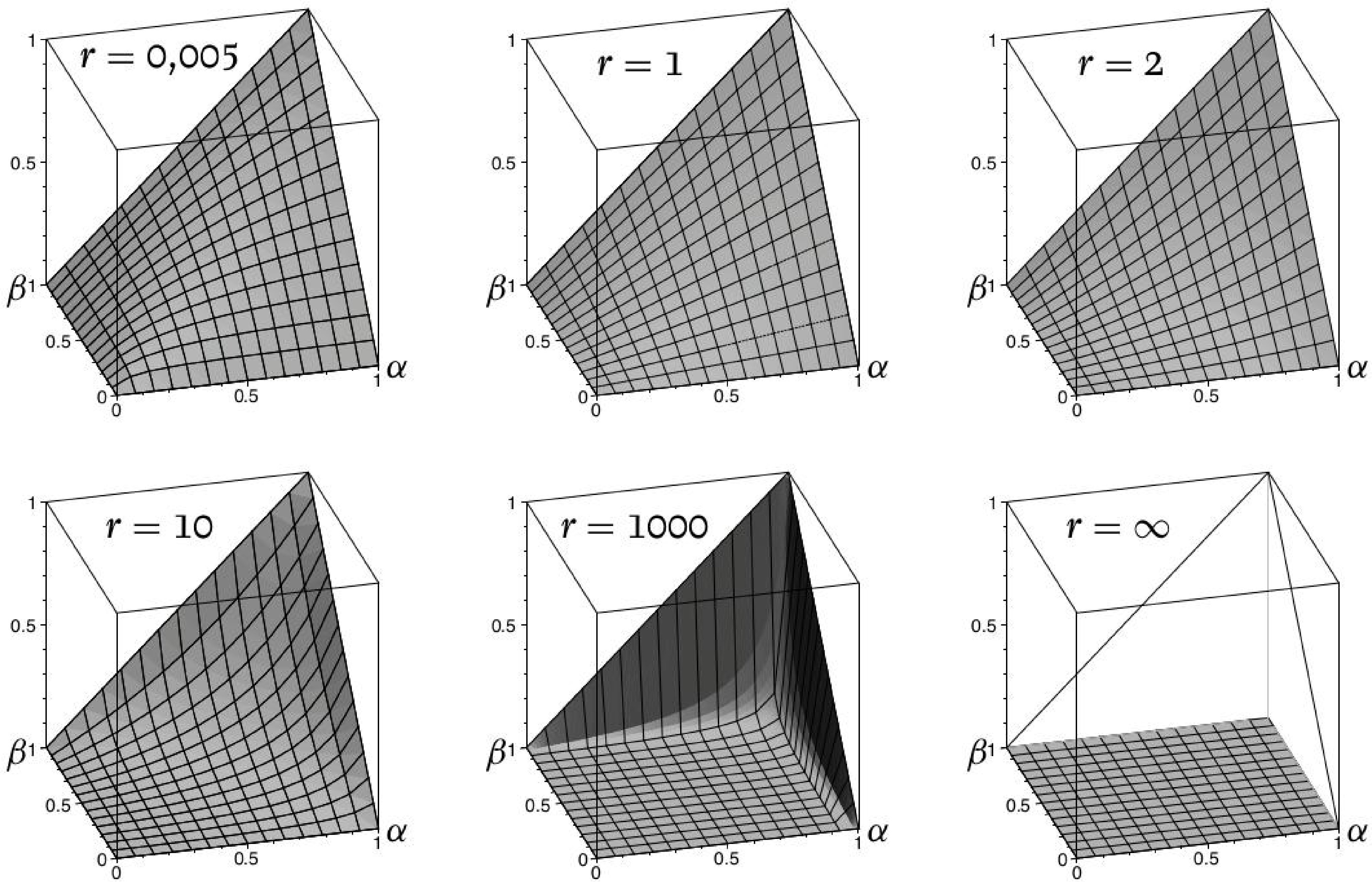

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$$\begin{aligned} i(\beta) &\leq i(\gamma), \\ i(\alpha) \underset{1}{\wedge} i(\beta) &\leq i(\alpha) \underset{1}{\wedge} i(\gamma), \\ \alpha \underset{2}{\wedge} \beta = i^{-1}(i(\alpha) \underset{1}{\wedge} i(\beta)) &\leq i^{-1}(i(\alpha) \underset{1}{\wedge} i(\gamma)) = \alpha \underset{2}{\wedge} \gamma. \end{aligned}$$

- Boundary condition:

$$\alpha \underset{2}{\wedge} 1 = i^{-1}(i(\alpha) \underset{1}{\wedge} i(1)) = i^{-1}(i(\alpha) \underset{1}{\wedge} 1) = i^{-1}(i(\alpha)) = \alpha.$$

Classification of fuzzy conjunctions

Continuous fuzzy conjunction $\overset{.}{\wedge}$ is

- **Archimedean** if
$$\forall \alpha \in (0,1): \quad \alpha \mathbin{\dot\wedge} \alpha < \alpha \tag{TA}$$
- **strict** if
$$\forall \alpha \in (0,1] \; \forall \beta, \gamma \in [0,1]: \quad \beta < \gamma \Rightarrow \alpha \mathbin{\dot\wedge} \beta < \alpha \mathbin{\dot\wedge} \gamma \tag{T3+}$$
- **nilpotent** if it is Archimedean and not strict.

Example: Product conjunction is strict, Łukasiewicz conjunction is nilpotent, standard and drastic conjunctions are not Archimedean (the standard one violates (TA), the drastic one is not continuous).

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Representation theorem for strict fuzzy conjunctions

Operation $\underset{\cdot}{\wedge} : [0,1]^2 \to [0,1]$ is a strict fuzzy conjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**multiplicative generator**) such that

$$\alpha \underset{\cdot}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{\mathrm{P}}{\wedge} i(\beta)\big) = i^{-1}\big(i(\alpha) \cdot i(\beta)\big).$$

Sufficiency has been already proved (except for strictness which is easy).
The proof of necessity is much more advanced.

A multiplicative generator of a strict fuzzy conjunction is not unique.

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Representation theorem for nilpotent fuzzy conjunctions

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Theorem: Let $\underset{\cdot}{\wedge}$ be a **nilpotent** fuzzy conjunction. Then

$$\forall \alpha \in (0,1)\ \exists n \in \mathbb{N} : \underset{\cdot}{\bigwedge}\nolimits_{k=1}^{n} \alpha = 0$$

Proof: According to the representation theorem, it suffices (without loss of generality) to prove the theorem for the Łukasiewicz conjunction. For a sufficiently large n we obtain

$$\alpha + \sum_{i=2}^{n} (\alpha - 1) \leq 0, \qquad \underset{\mathrm{L}}{\bigwedge}\nolimits_{k=1}^{n} \alpha = 0.$$

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Representation theorem for nilpotent fuzzy conjunctions

Operation $\dot\wedge : [0,1]^2 \to [0,1]$ is a **nilpotent** fuzzy conjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**Łukasiewicz generator**) such that

$$\alpha \mathbin{\dot\wedge} \beta = i^{-1}\big(i(\alpha) \mathbin{\underset{\text{L}}{\wedge}} i(\beta)\big).$$

A Łukasiewicz generator of a nilpotent fuzzy conjunction is not unique.

Theorem: Let $\dot\wedge$ be a **nilpotent** fuzzy conjunction. Then

$$\forall \alpha \in (0,1)\ \exists n \in \mathbb{N} : \dot{\bigwedge}_{k=1}^{n} \alpha = 0$$

Proof: According to the representation theorem, it suffices (without loss of generality) to prove the theorem for the Łukasiewicz conjunction. For a sufficiently large n we obtain

$$\alpha + \sum_{i=2}^{n} (\alpha - 1) \leq 0, \qquad \underset{\text{L}}{\bigwedge}_{k=1}^{n} \alpha = 0.$$

Fuzzy intersection

is an operation on fuzzy sets defined using a fuzzy conjunction:

$$\mu_{A \underset{\cdot}{\cap} B}(x) = \mu_A(x) \underset{\cdot}{\wedge} \mu_B(x)$$

(we distinguish them by the same indices as the respective fuzzy conjunctions)

Theorem: The **standard** intersection is cut-consistent.

Proof: 1. Cutworthiness:

$$\begin{aligned} \mathcal{R}_{A \underset{S}{\cap} B}(\alpha) &= \{x \in X : \mu_{A \underset{S}{\cap} B}(x) \geq \alpha\} \\ &= \{x \in X : (\mu_A(x) \geq \alpha) \wedge (\mu_B(x) \geq \alpha)\} \\ &= \{x \in X : \mu_A(x) \geq \alpha\} \cap \{x \in X : \mu_B(x) \geq \alpha\} \\ &= \mathcal{R}_A(\alpha) \cap \mathcal{R}_B(\alpha) \end{aligned}$$

2. Cuts $\mathcal{R}_A(\alpha) \cap \mathcal{R}_B(\alpha)$ (for all $\alpha \in (0, 1]$) determine a unique fuzzy set equal to $A \underset{S}{\cap} B$.