Fuzzy relations

A **fuzzy relation** is $R \in \mathcal{F}(X \times Y)$, $\mu_R : X \times Y \to [0, 1]$

The **inverse relation** to R is $R^{-1} \in \mathcal{F}(Y \times X)$:

$$\forall x \in X \ \forall y \in Y : \mu_{R^{-1}}(y, x) = \mu_R(x, y)$$

The $\dot{}$-**composition** of relations $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ is $R \underset{\cdot}{\circ} S \in \mathcal{F}(X \times Z)$:

$$\mu_{R \underset{\cdot}{\circ} S}(x, z) = \sup_{y \in Y} \big(\mu_R(x, y) \underset{\cdot}{\wedge} \mu_S(y, z)\big)$$

Theorem The inversion of fuzzy relations is cut-consistent.

Theorem If Y is a finite set, then the standard composition of fuzzy relations $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ is cut-consistent.

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Special crisp relations

$R \subseteq X \times X$ can be:

- **an equality**: $E = \{(x,x) : x \in X\}$,
- **reflexive**: $\forall x \in X : (x,x) \in R$, i.e., $E \subseteq R$,
- **symmetric**: $(x,y) \in R \Rightarrow (y,x) \in R$, i.e., $R = R^{-1}$,
- **antisymmetric**: $\big((x,y) \in R\big) \wedge \big((y,x) \in R\big) \Rightarrow x = y$, i.e., $R \cap R^{-1} \subseteq E$,
- **transitive**: $\big((x,y) \in R\big) \wedge \big((y,z) \in R\big) \Rightarrow (x,z) \in R$, i.e., $R \circ R \subseteq R$,
- **a partial order**: antisymmetric, reflexive, and transitive,
- **an equivalence**: symmetric, reflexive, and transitive.

The membership function of the equality relation, $E \subseteq X \times X$, is the **Kronecker delta**:

$$\mu_E(x,y) = \delta(x,y) = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{for } x \neq y. \end{cases}$$

Special fuzzy relations

A fuzzy relation $R \in \mathcal{F}(X \times X)$ can be:

- **reflexive**: $E \subseteq R$,
- **symmetric**: $R = R^{-1}$,
- **$\cdot$-antisymmetric**: $R \underset{\cdot}{\cap} R^{-1} \subseteq E$,
- **$\cdot$-transitive**: $R \underset{\cdot}{\circ} R \subseteq R$,
- **a $\cdot$-partial order**: $\cdot$-antisymmetric, reflexive, and $\cdot$-transitive,
- **an $\cdot$-equivalence**: symmetric, reflexive, and $\cdot$-transitive.

The last four terms depend on the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$.

Theorem The following properties of fuzzy relations are cut-consistent:

- reflexivity,
- symmetry,
- standard antisymmetry,
- product antisymmetry,
- standard transitivity,
- standard partial order,
- standard equivalence.

Projections of fuzzy relations

The **left (first) projection** of $R \in \mathcal{F}(X \times Y)$ is $P_1(R) \in \mathcal{F}(X)$: $\mu_{P_1(R)}(x) = \sup_{y \in Y} \mu_R(x, y)$.

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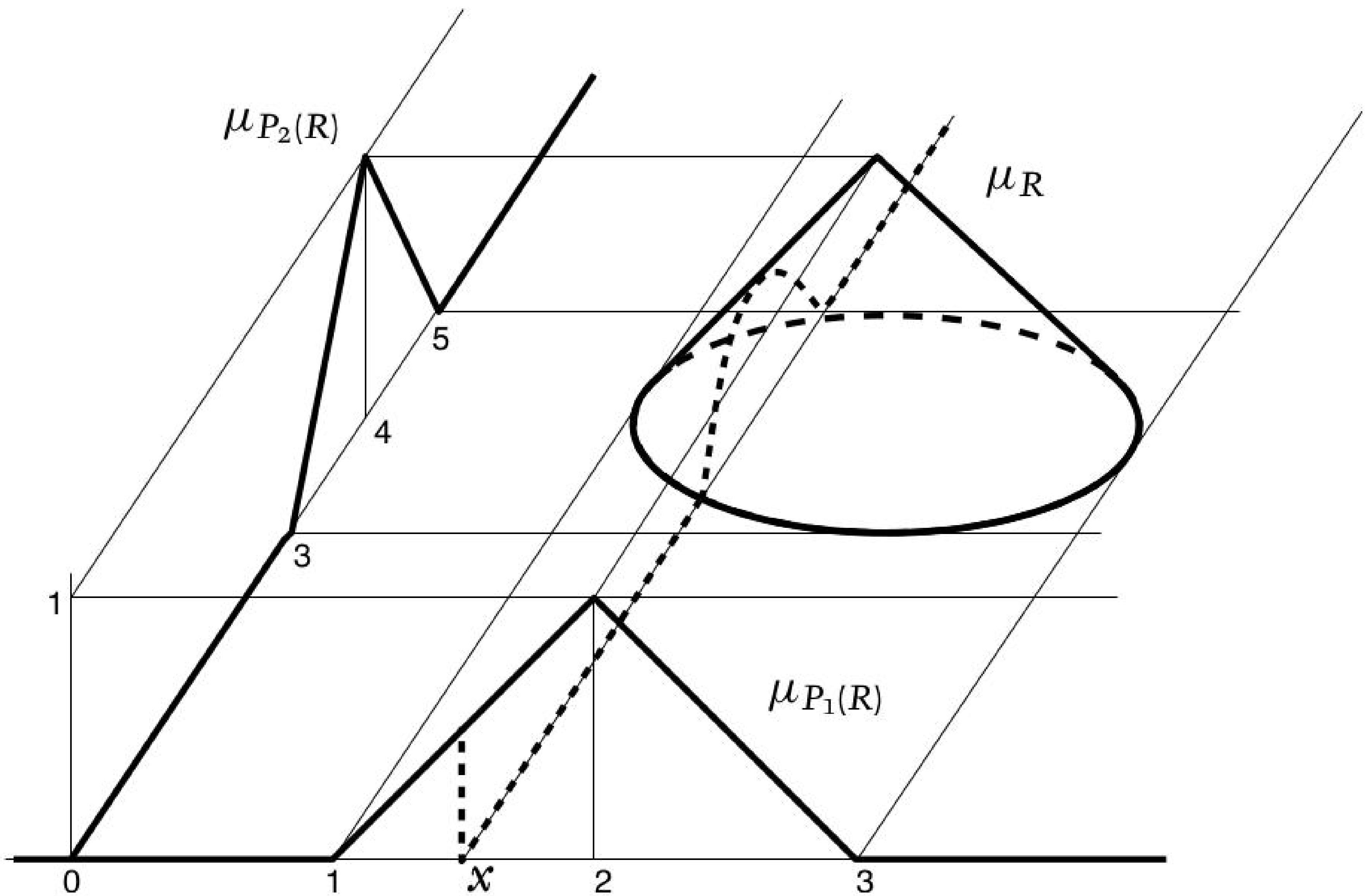

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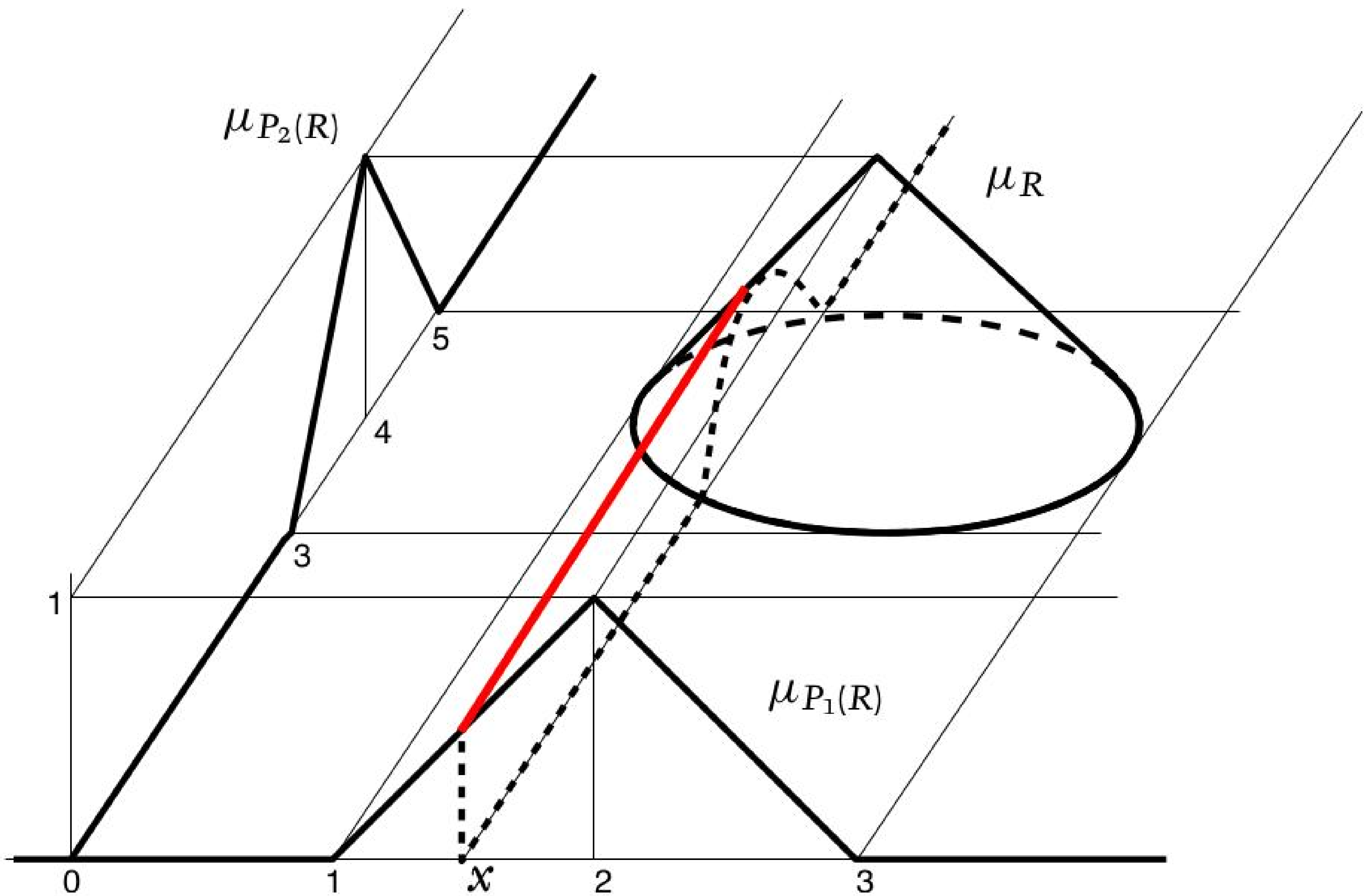

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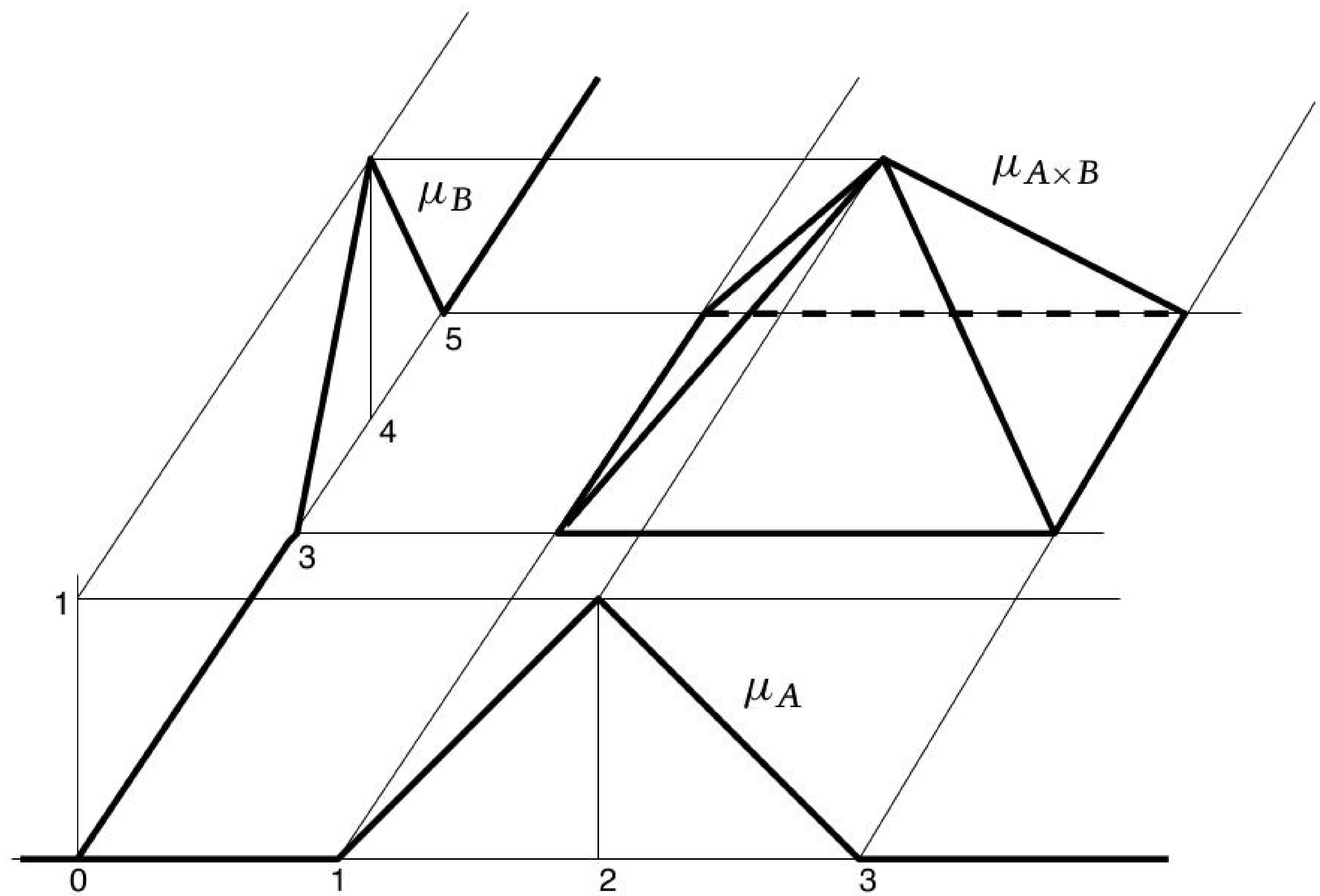

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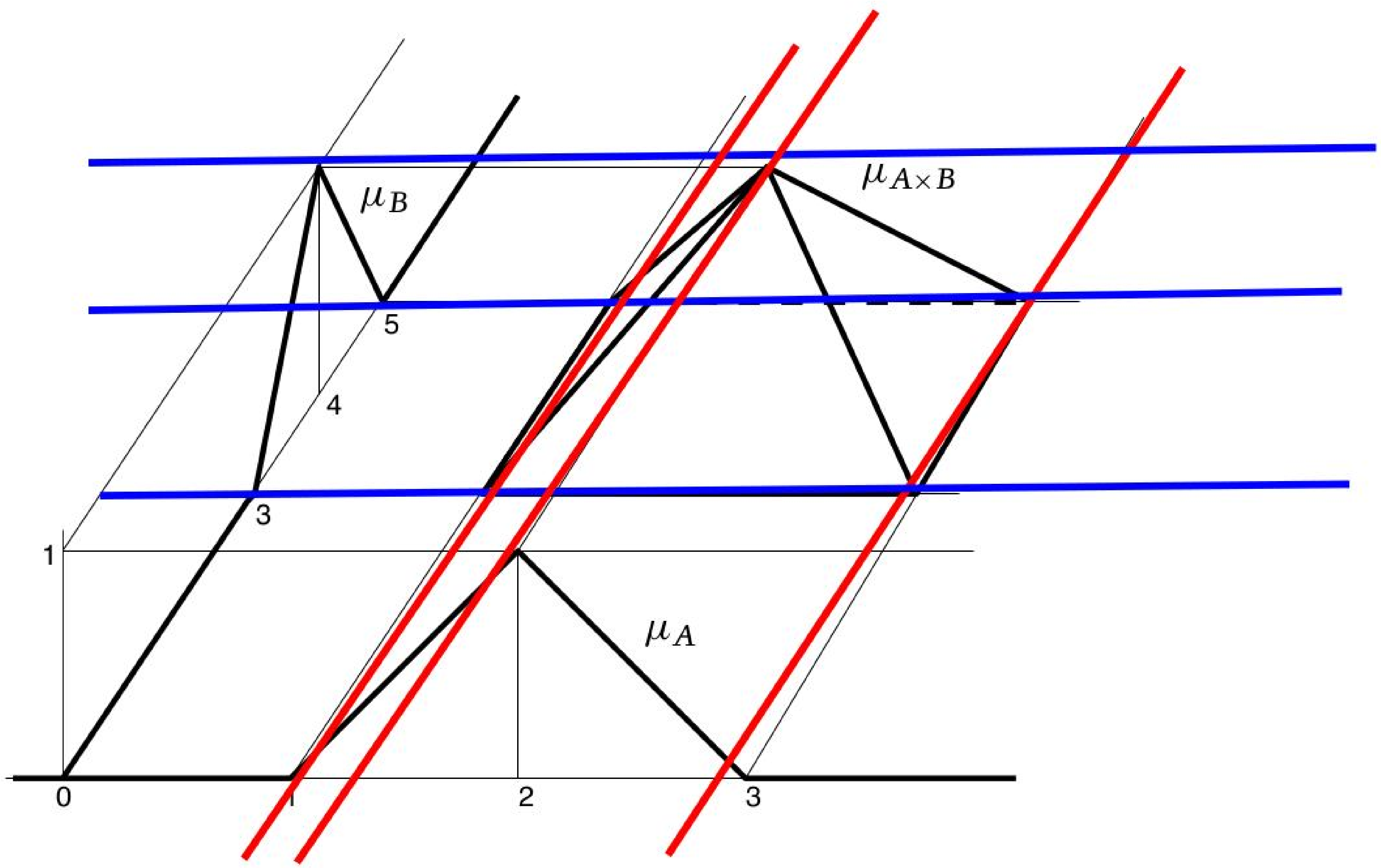

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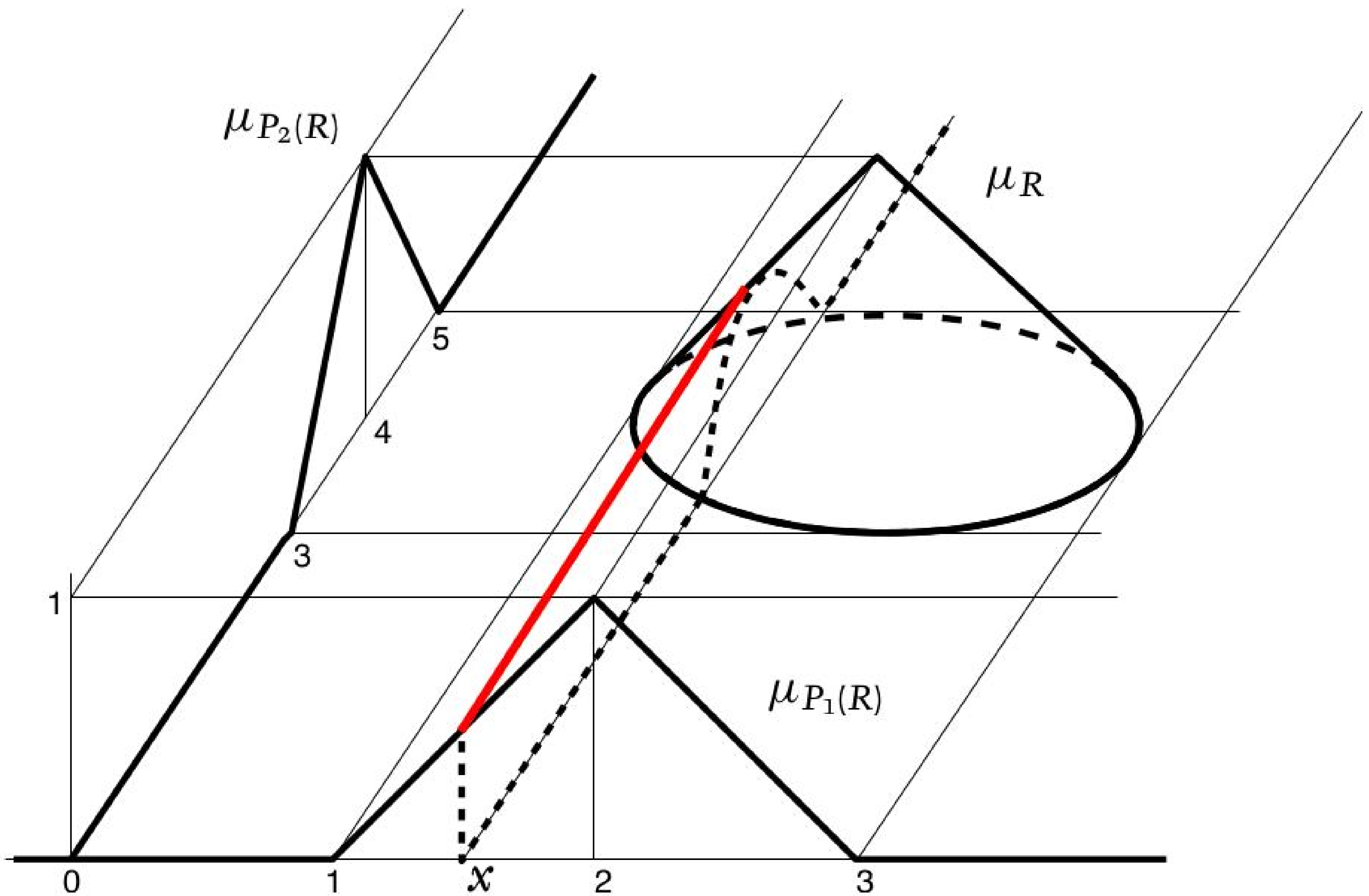

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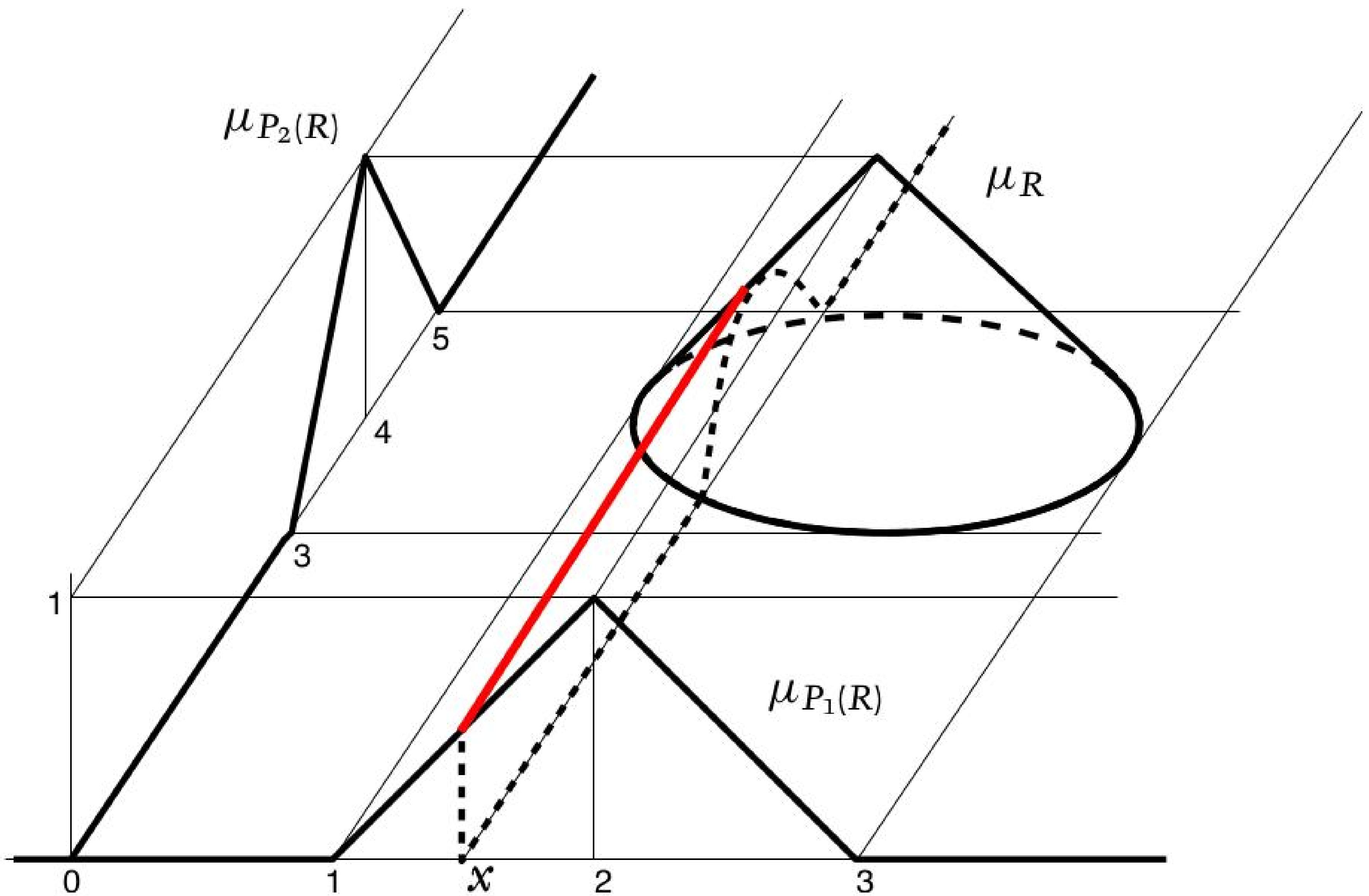

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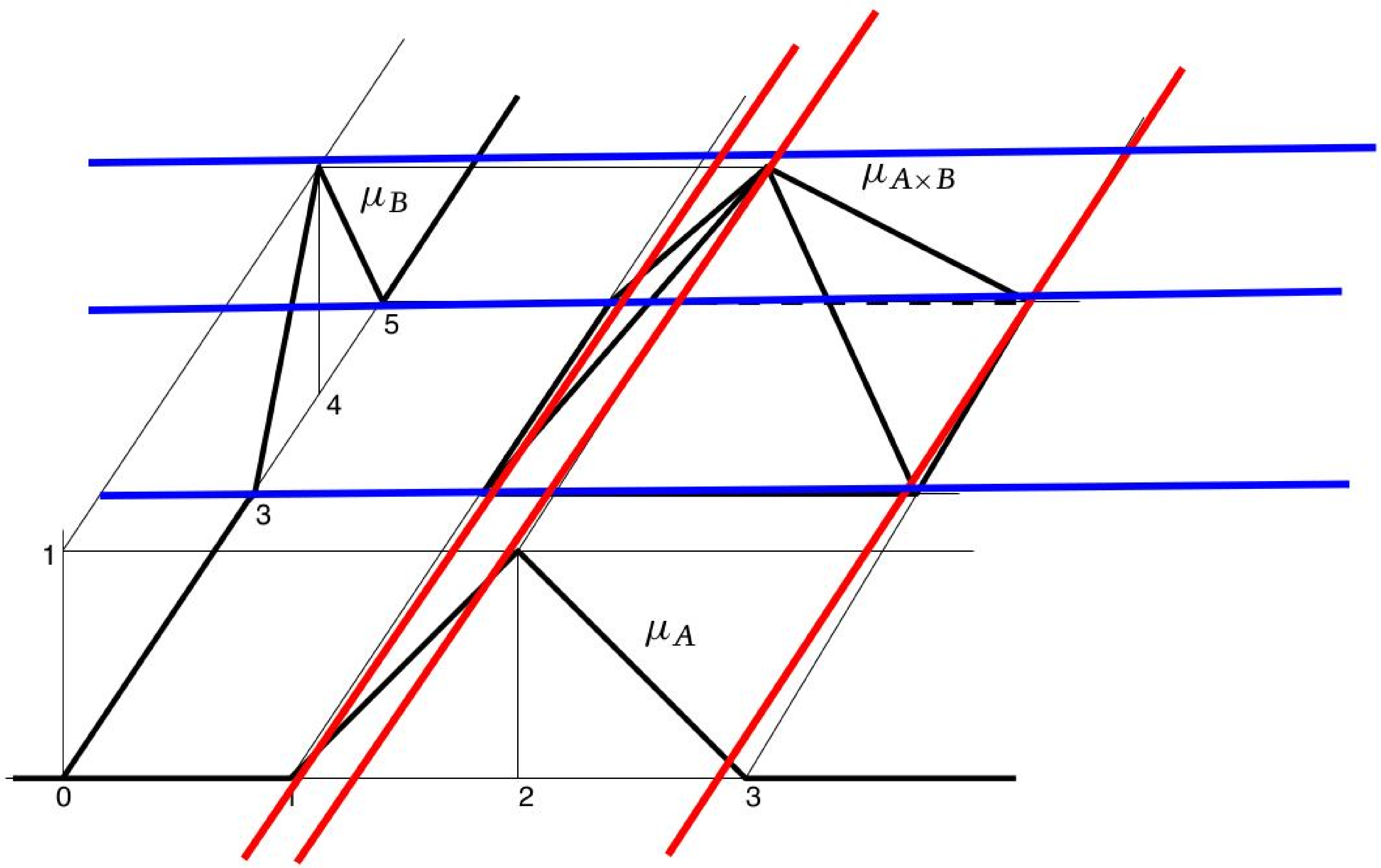

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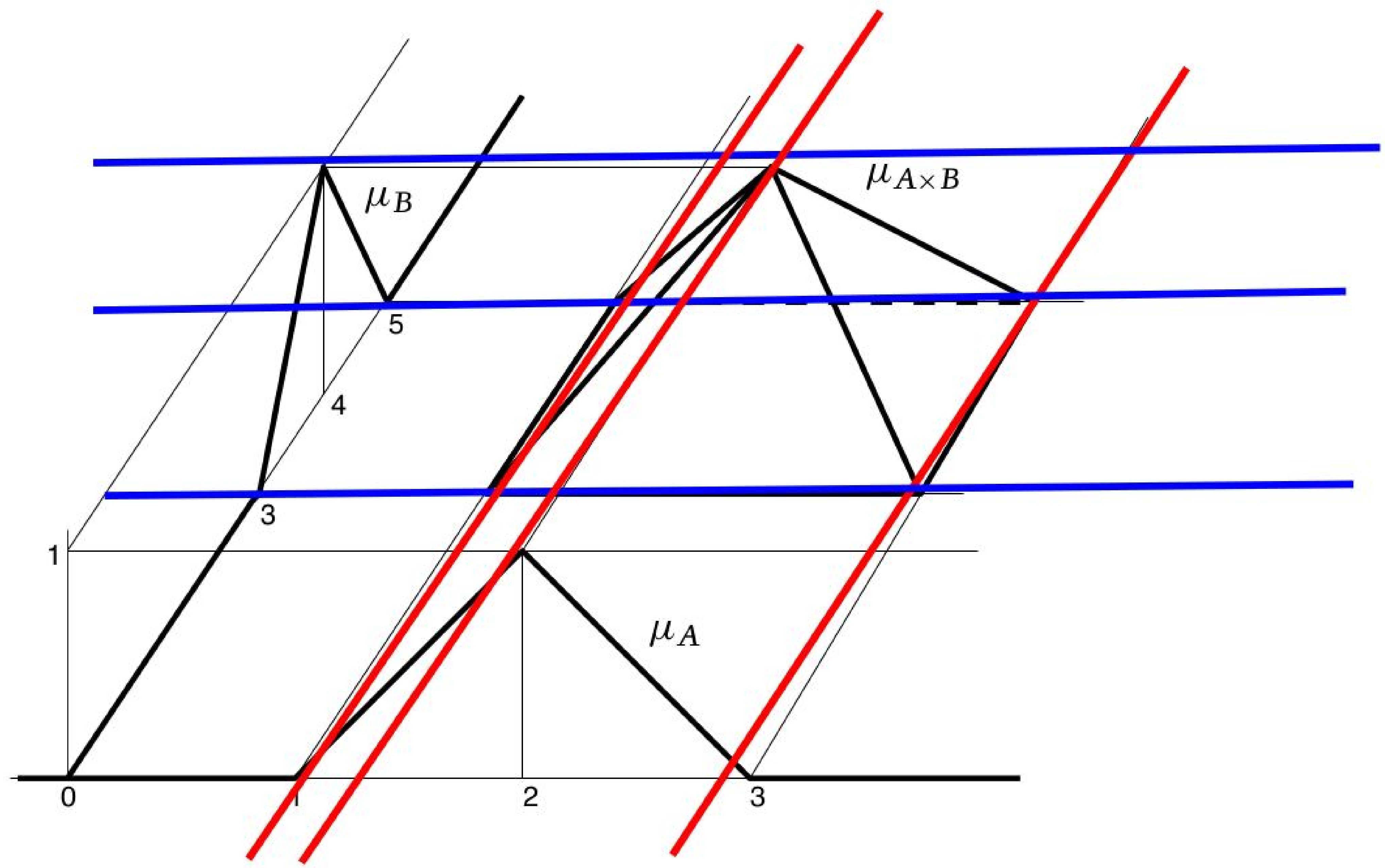

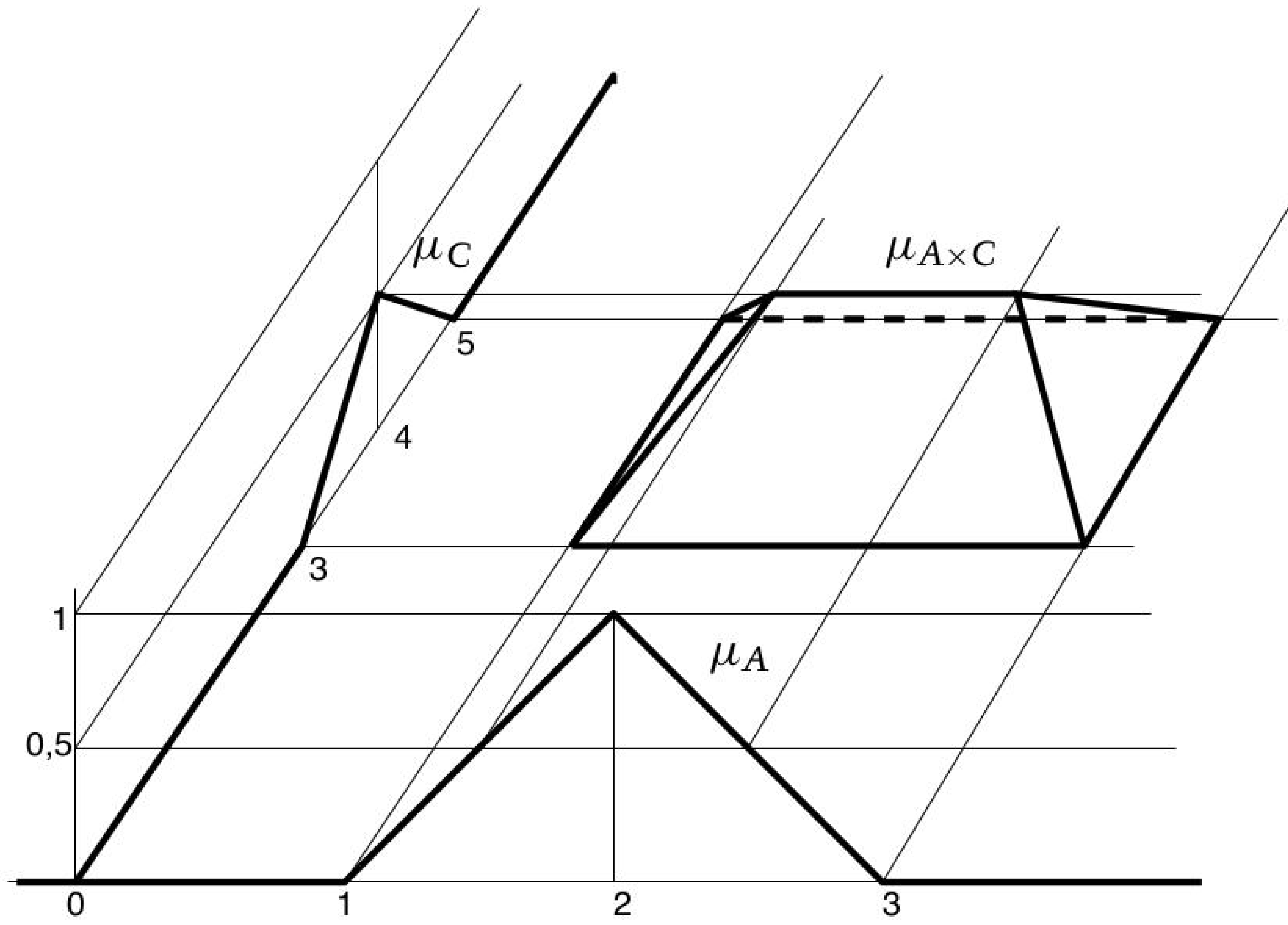

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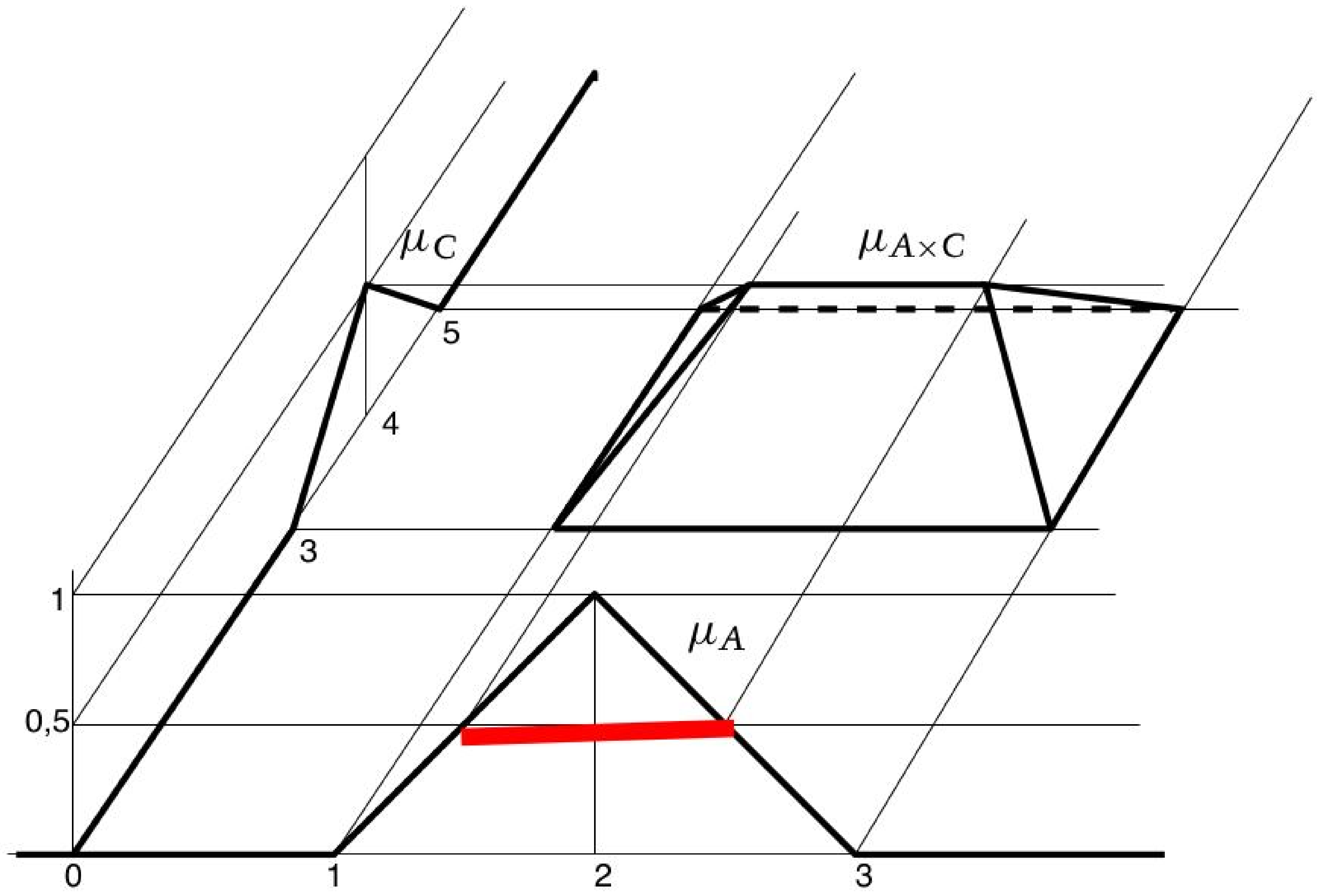

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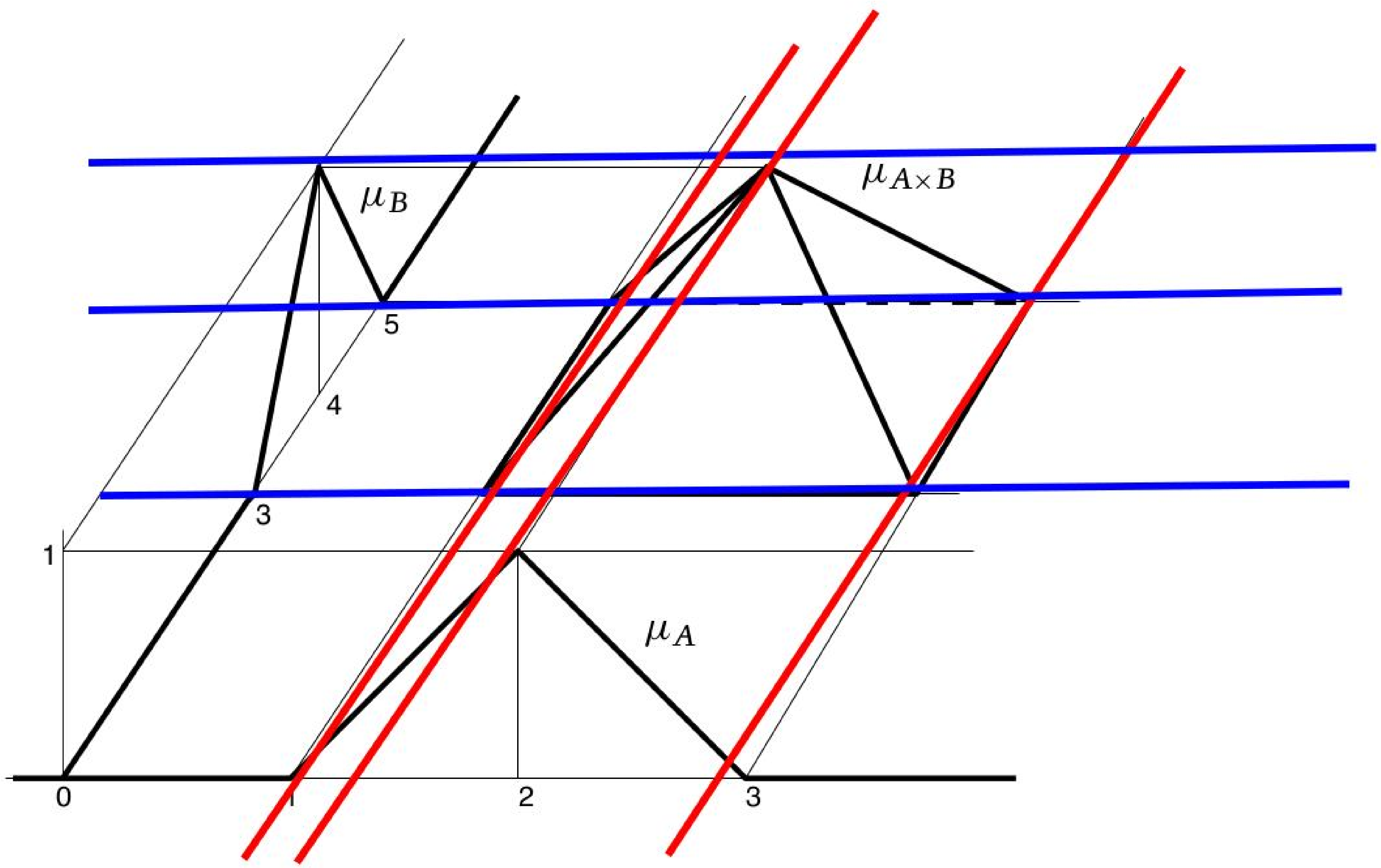

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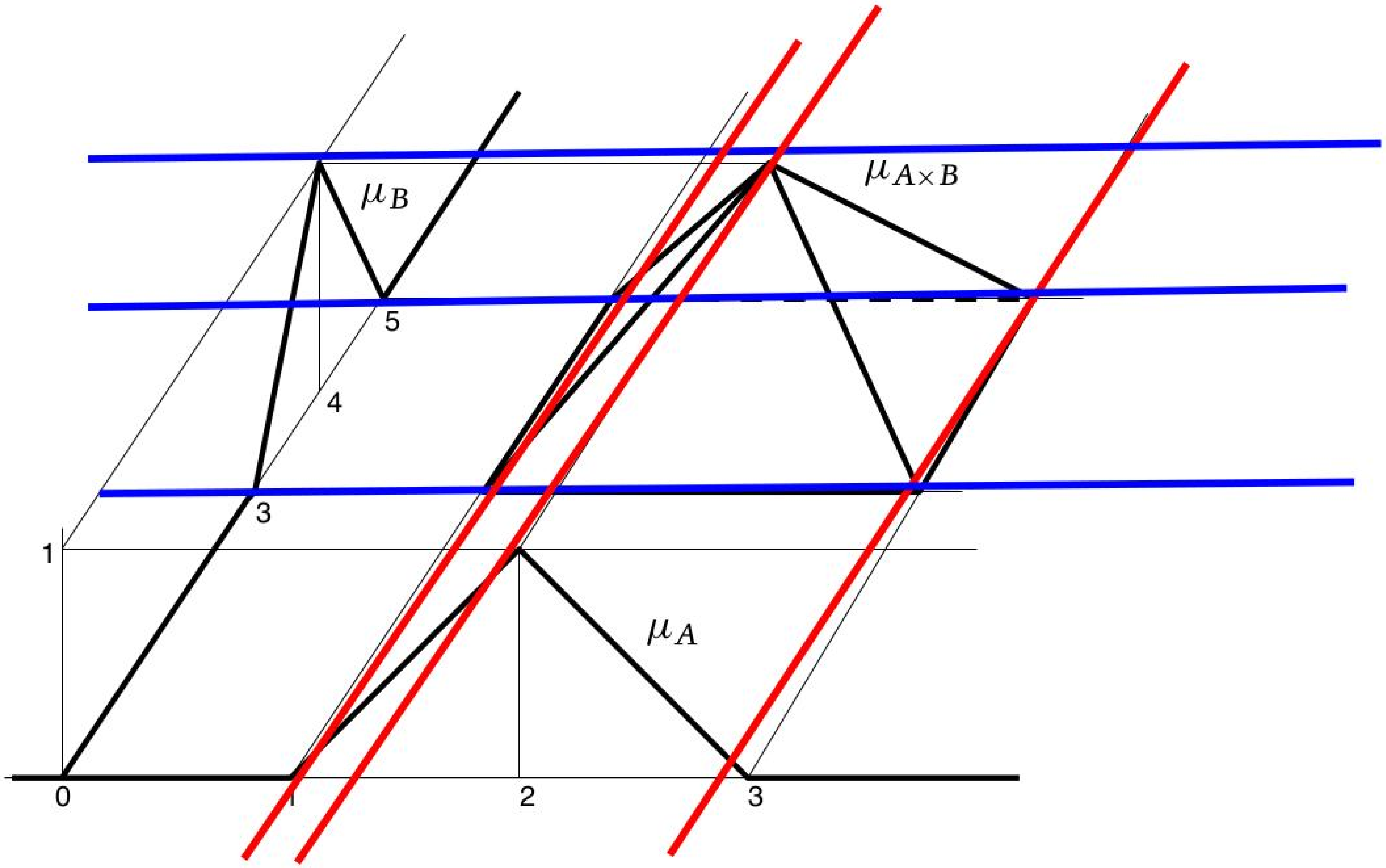

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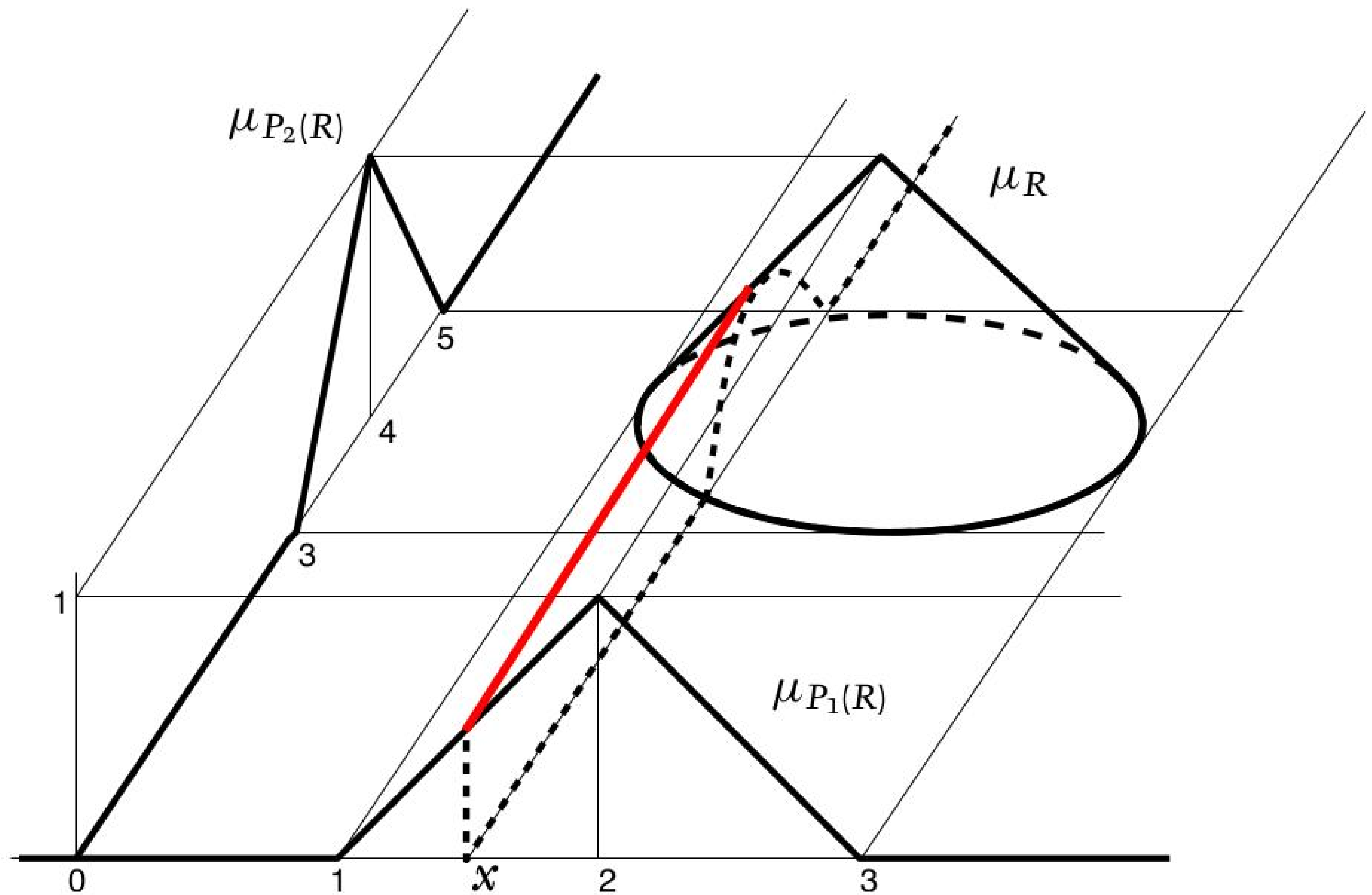

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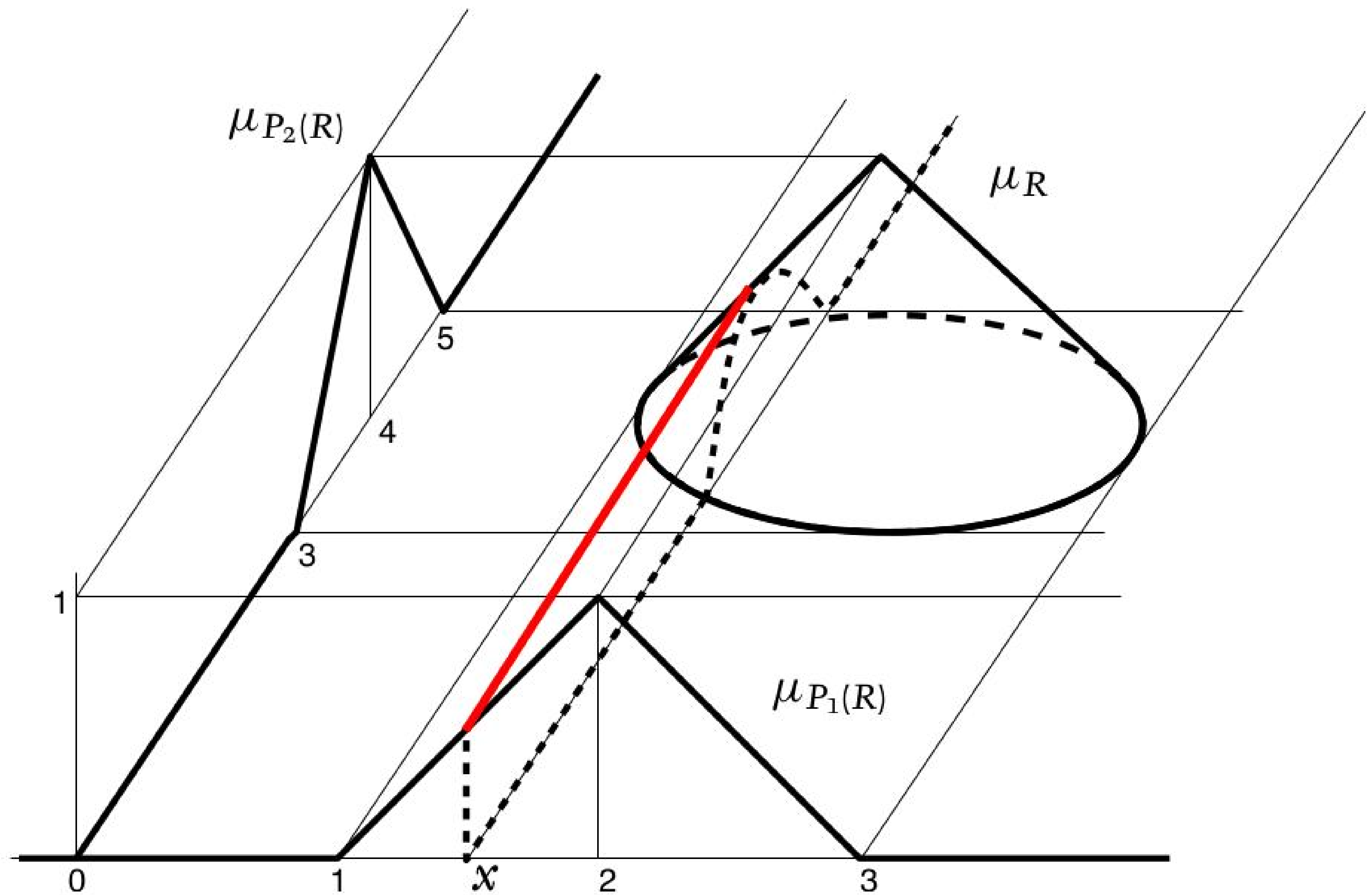

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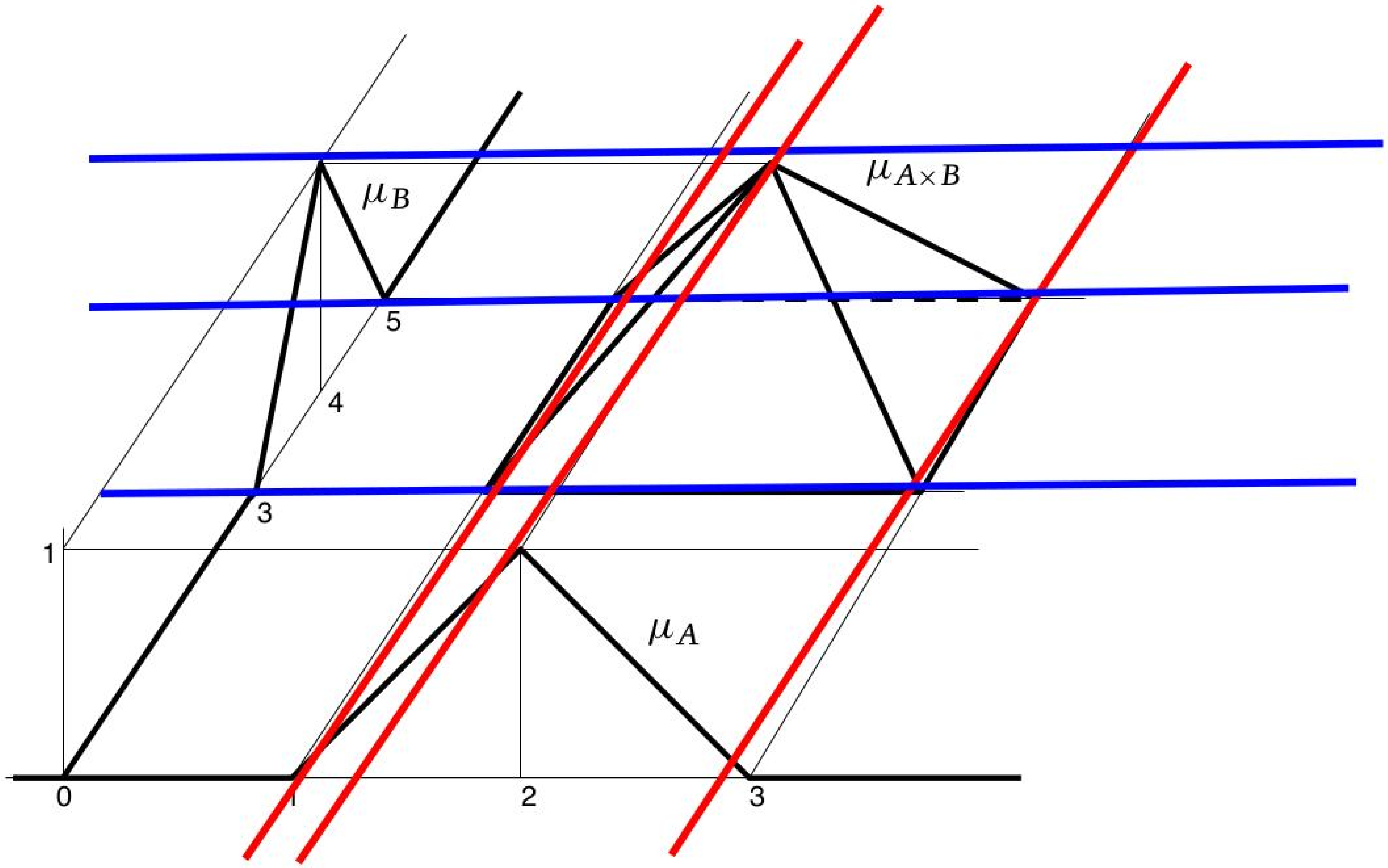

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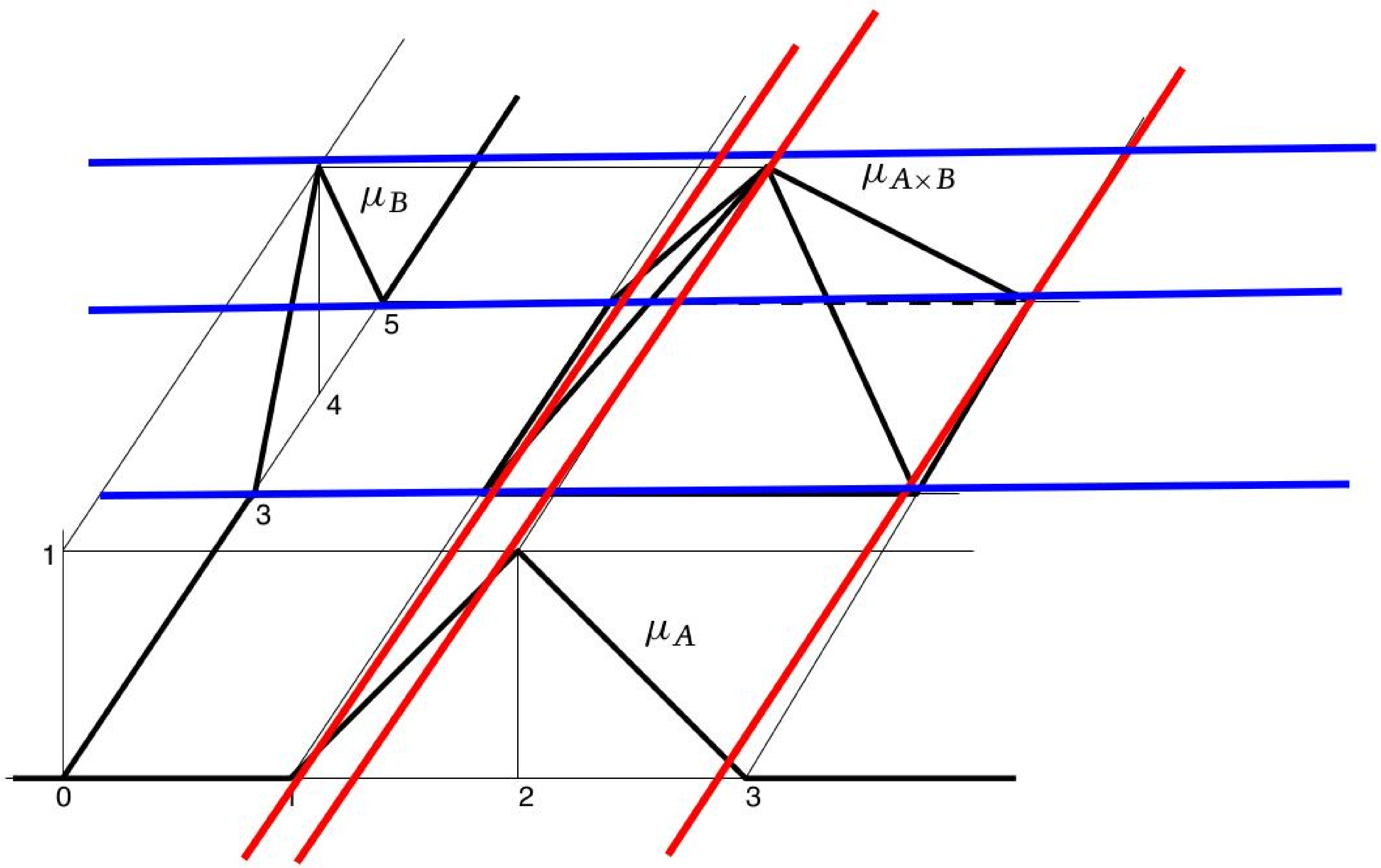

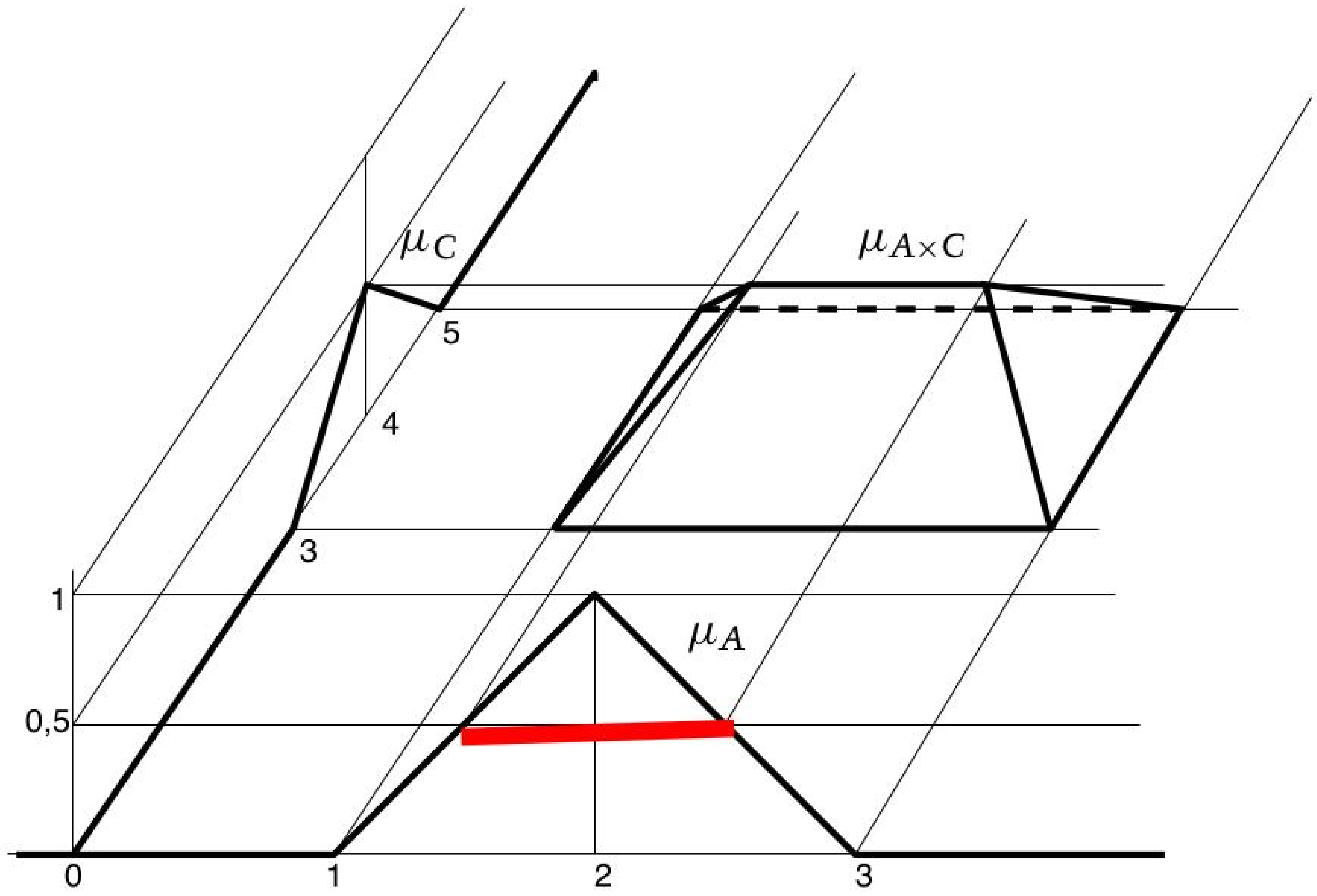

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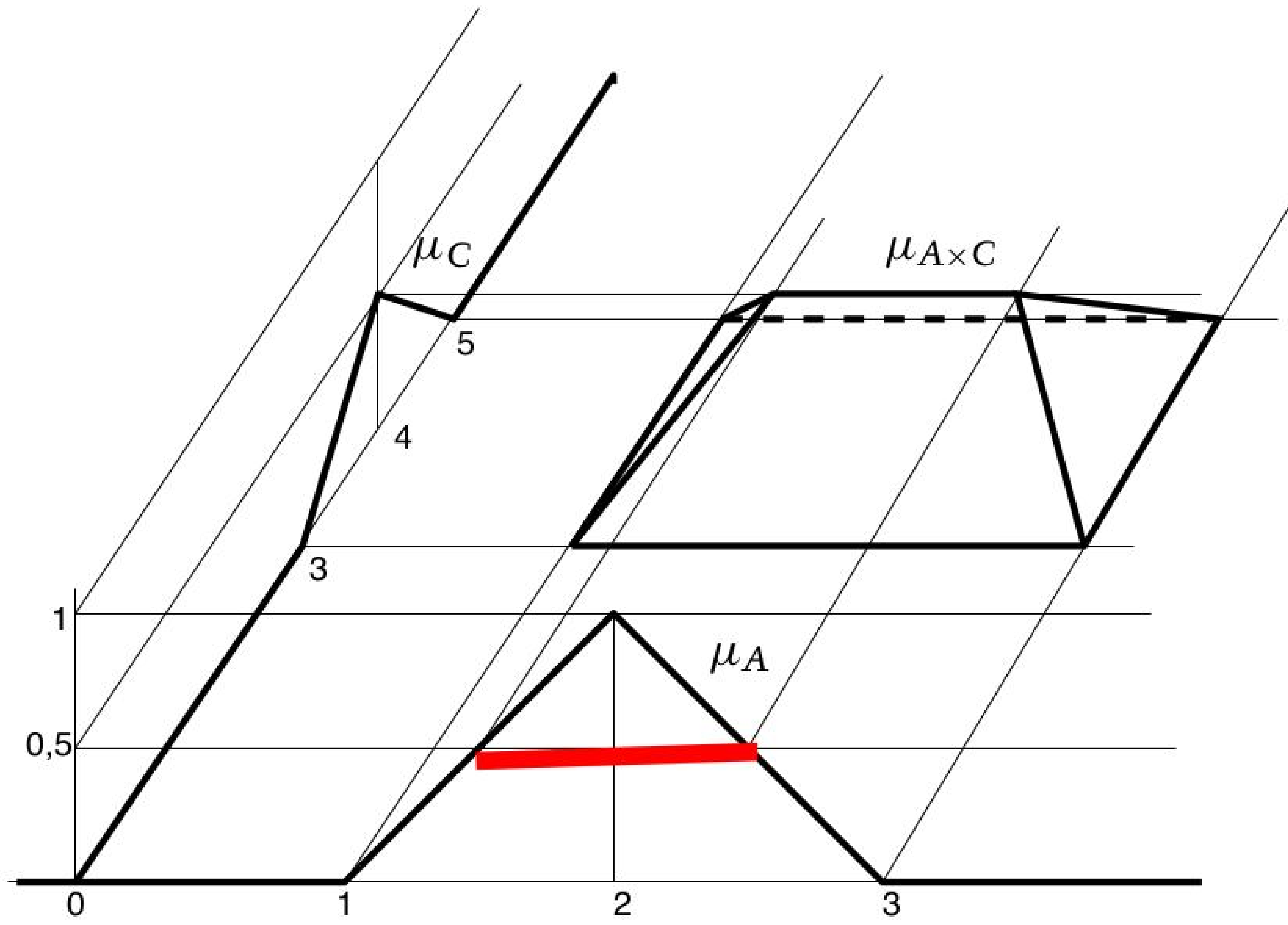

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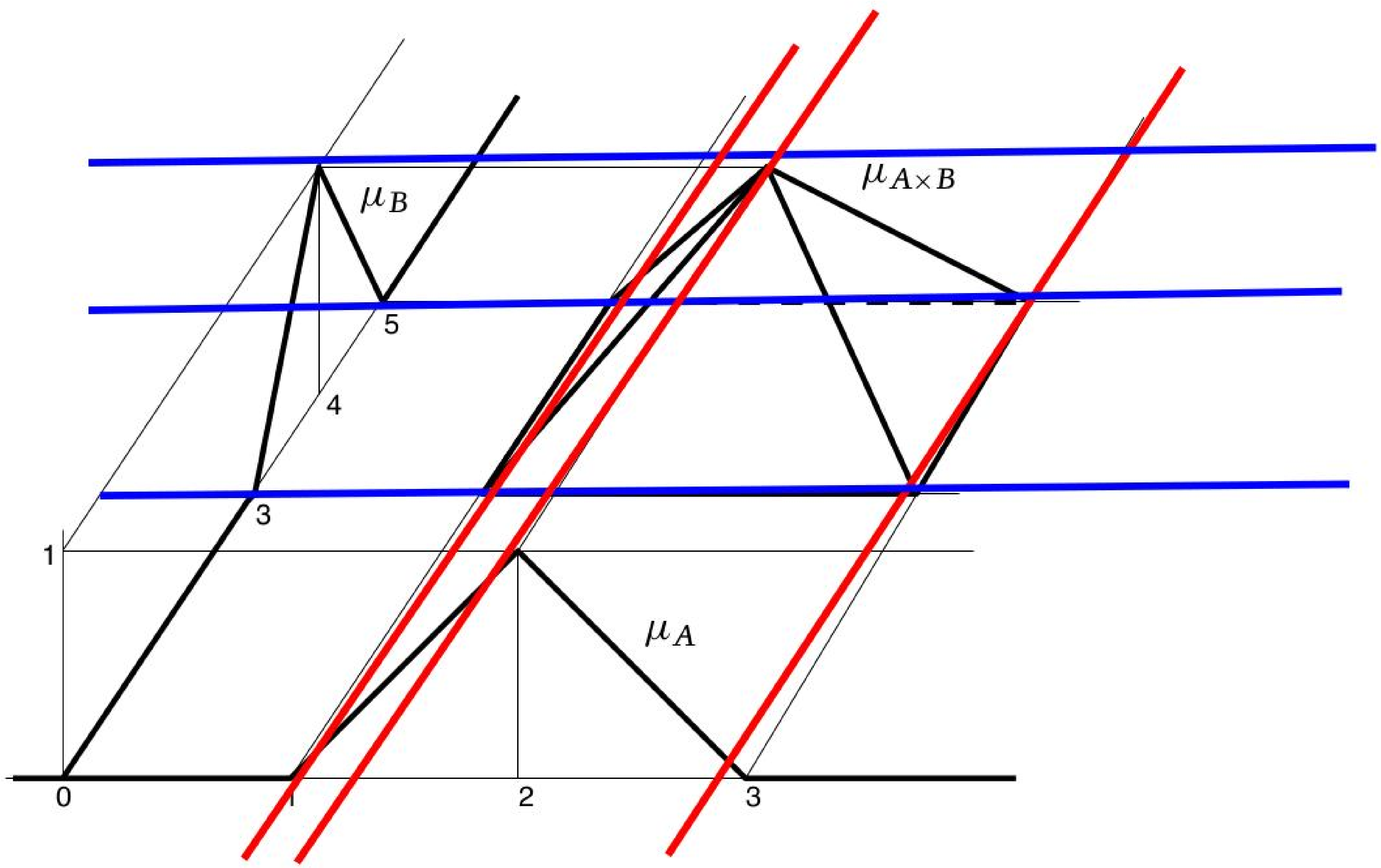

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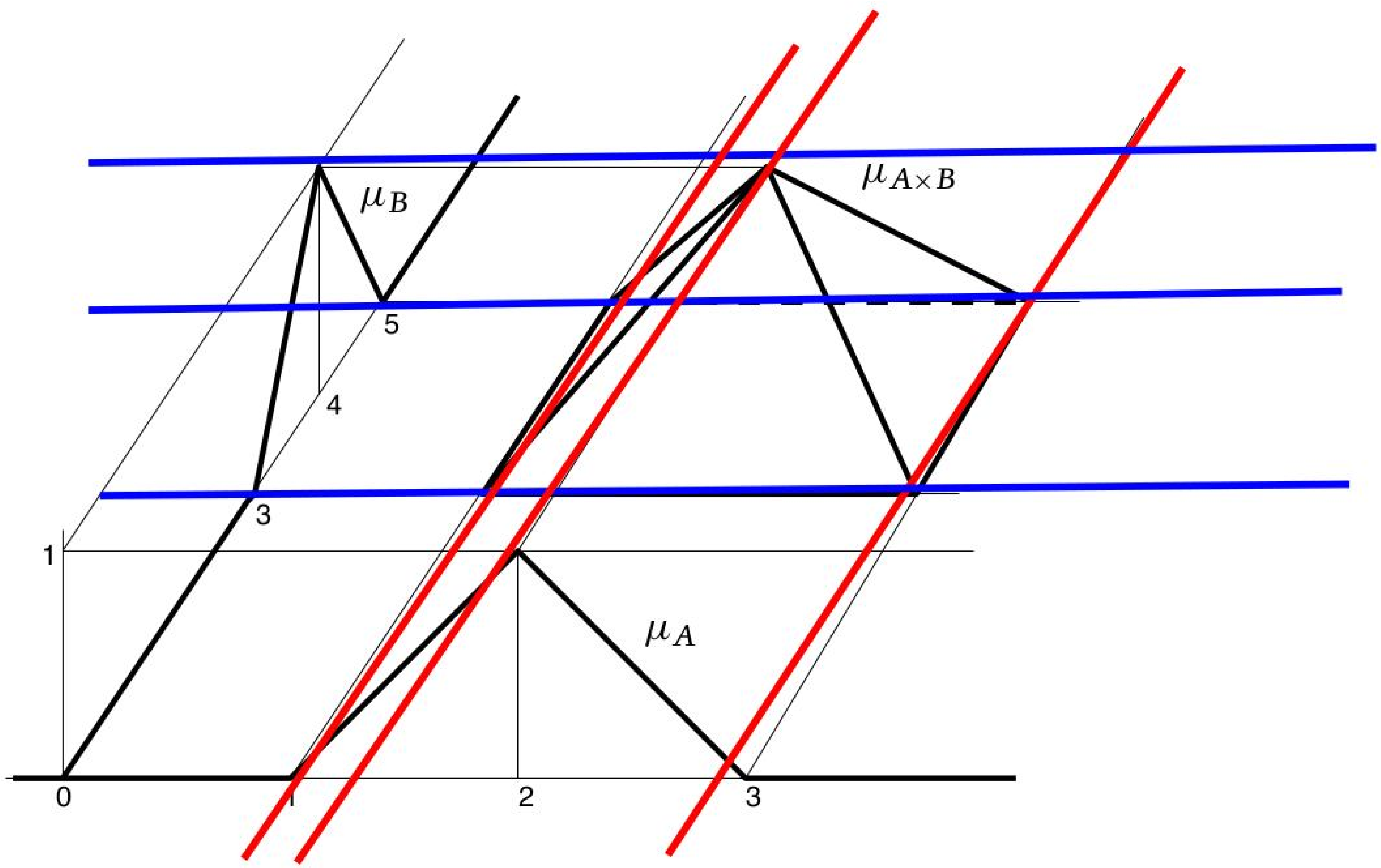

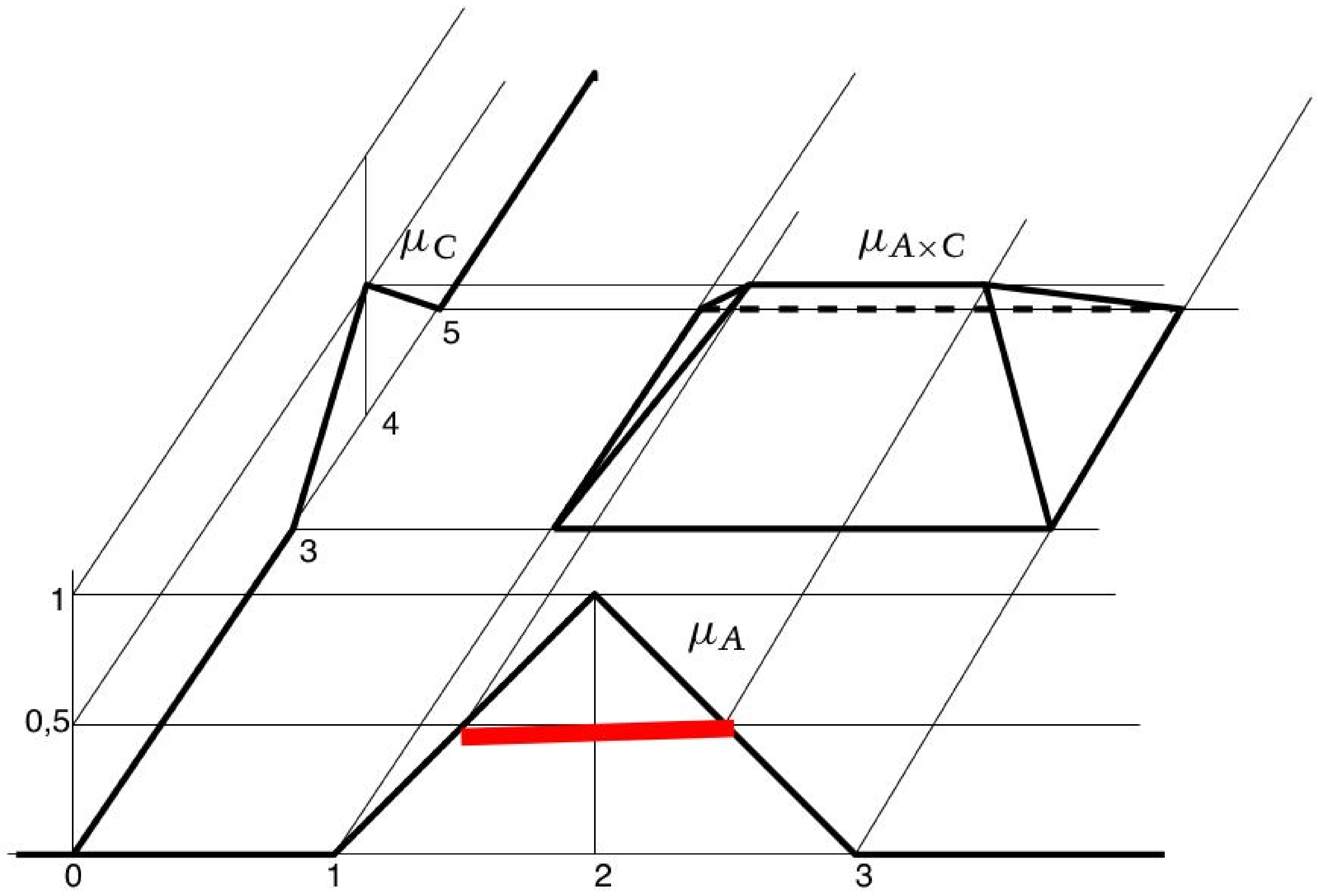

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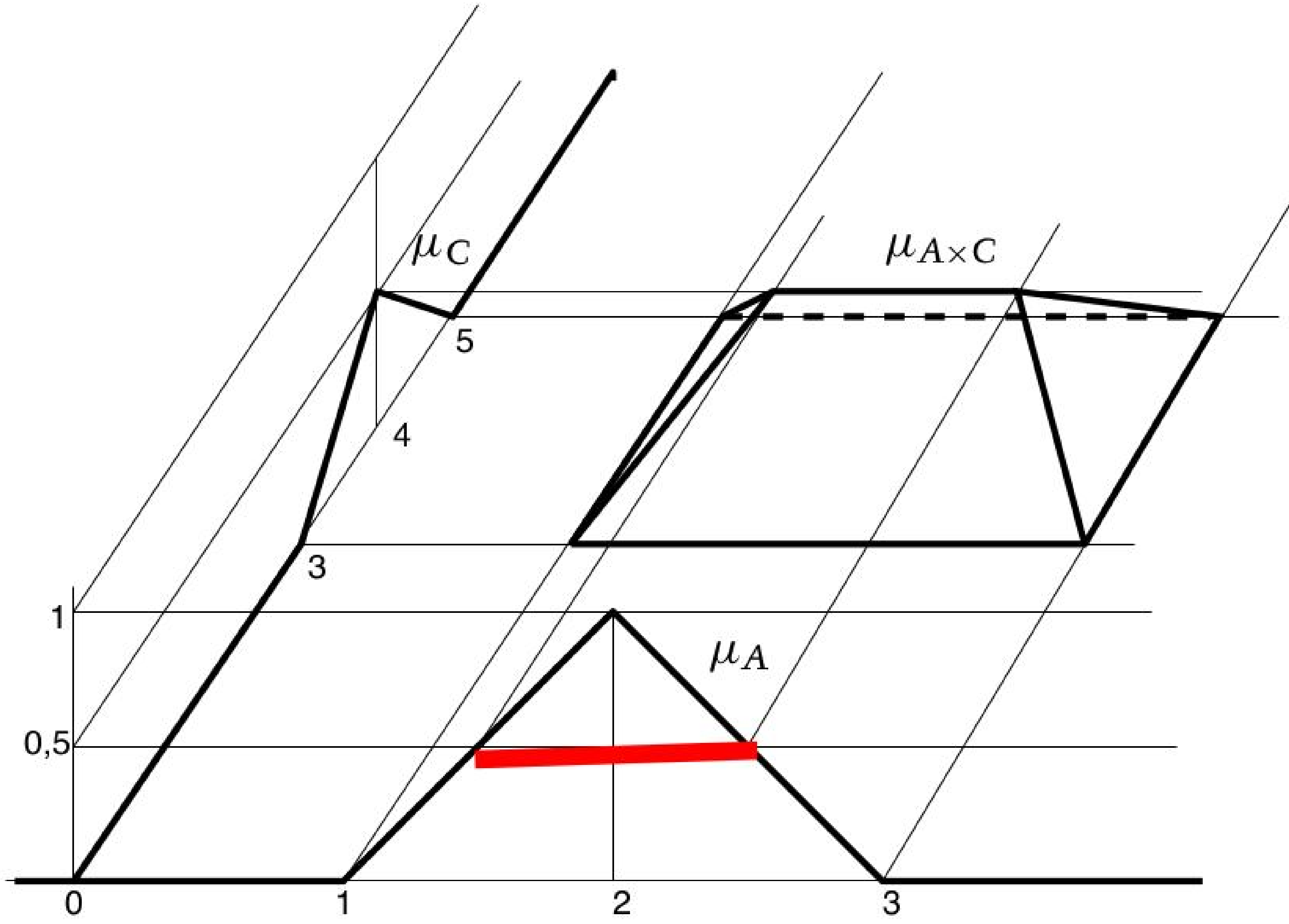

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The extension of binary relations to crisp sets

A **mapping** is $R \subseteq X \times Y$:

$$\forall x \in X \ \exists! y = r(x) \in Y : (x, y) \in R$$

A mapping $R \subseteq X \times Y$ corresponds to an $r : X \to Y$ by $(x, y) \in R \iff y = r(x)$,
$R = \{ (x, r(x)) : x \in X \}$

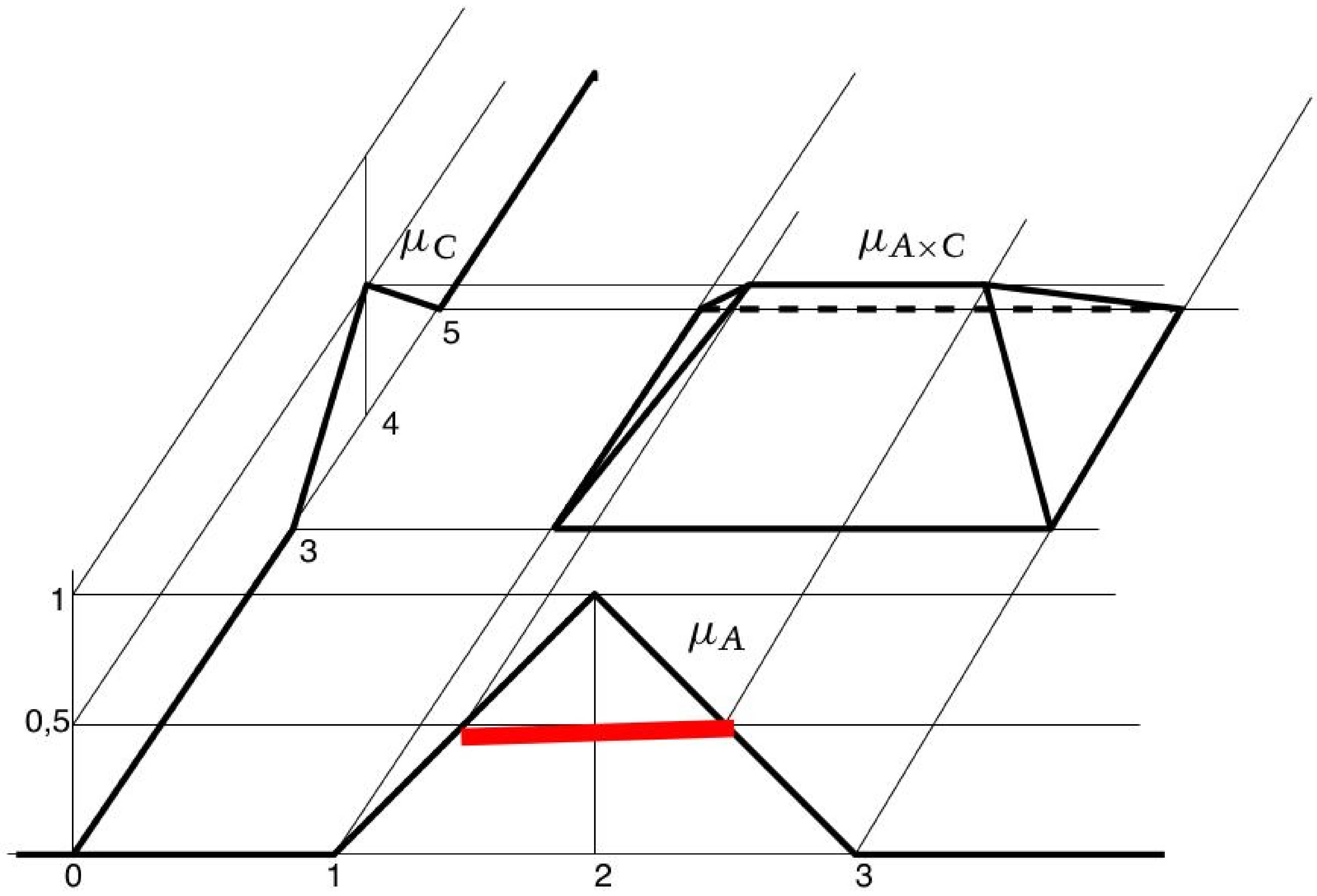
The **extension** of a relation $R \subseteq X \times Y$ is a mapping
$r : \mathcal{P}(X) \to \mathcal{P}(Y)$:

$$r(A) = \{ y \in Y : \ (\exists x \in A : (x, y) \in R) \}$$

Analogously, the extension of the relation $R^{-1} \subseteq Y \times X$ is a mapping $r^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$:

$$r^{-1}(B) = \{ x \in X : \ (\exists y \in B : (x, y) \in R) \}$$

The extensions r and r^{-1} are mappings even if the original relation R was not a mapping. However, they are not mutually inverse.



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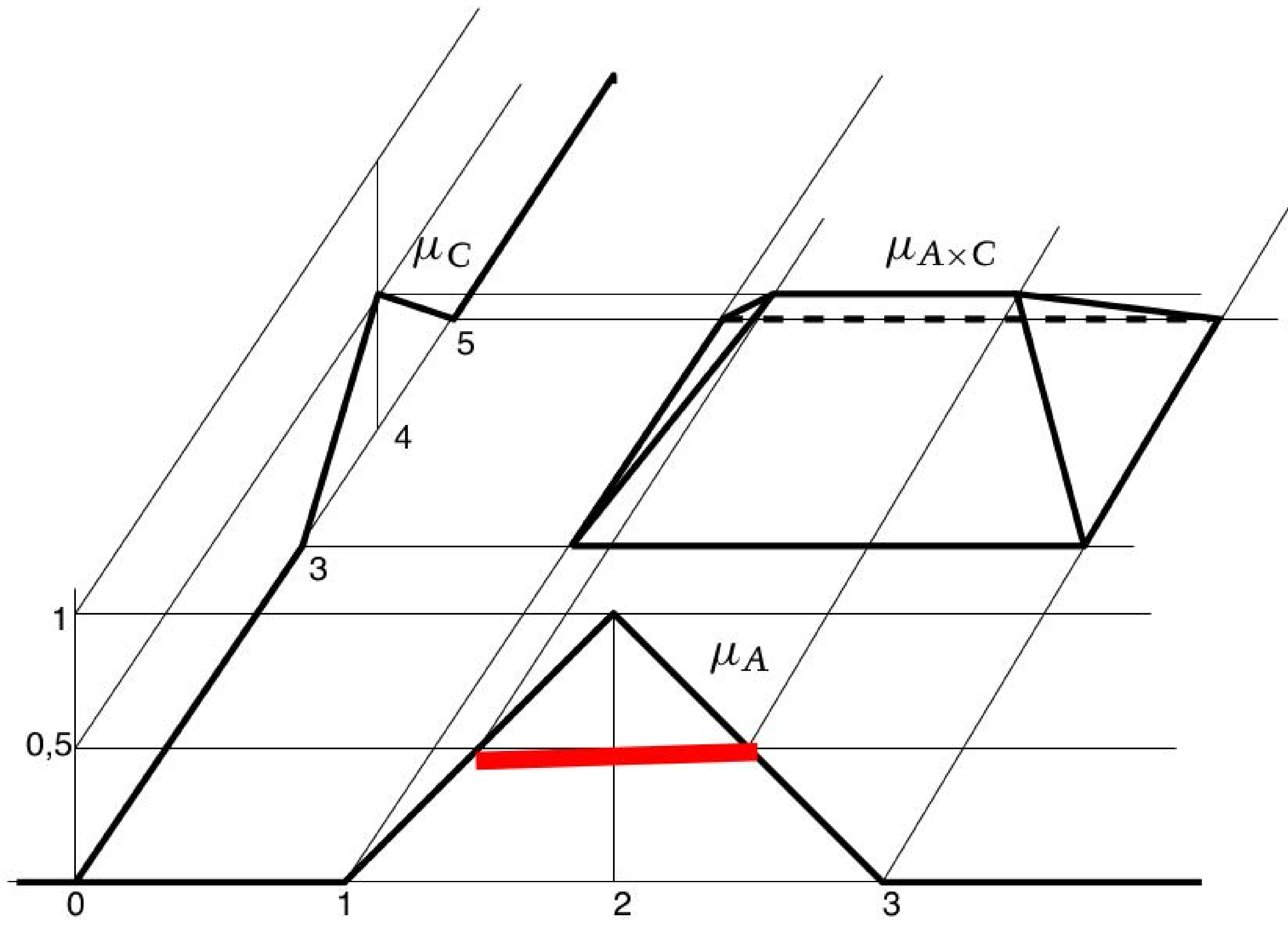

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The extensions r and r^{-1} are mappings even if the original relation R was not a mapping. However, they are not mutually inverse.

If, moreover, R is a mapping, then

$$\begin{aligned} r(A) &= \{r(x) : x \in A\} \\ r^{-1}(B) &= \{x \in X : r(x) \in B\} \end{aligned}$$

In particular,

$$r^{-1}(y) = r^{-1}(\{y\}) = \{x \in X : r(x) = y\}$$

Using membership functions:

$$\mu_{r(A)}(y) = \max_{x \in X}\big(\mu_R(x, y) \wedge \mu_A(x)\big)$$

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As R is a crisp relation, the choice of the fuzzy conjunction $\dot{\wedge}$ is irrelevant:

$$\mu_R(x,y) \mathbin{\underset{\cdot}{\wedge}} \mu_A(x) = \begin{cases} \mu_A(x) & \text{for } \mu_R(x,y) = 1 \\ 0 & \text{for } \mu_R(x,y) = 0 \end{cases}$$

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Analogously, the extension of the relation $R^{-1} \subseteq Y \times X$ is a mapping $r^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X)$:

$$\mu_{r^{-1}(B)}(x) = \sup_{y \in Y} \left(\mu_R(x, y) \underset{\mathrm{S}}{\wedge} \mu_B(y) \right) \qquad (B \in \mathcal{F}(Y),\ x \in X)$$

As R is a crisp relation, the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$ is irrelevant:

$$\mu_R(x, y) \underset{\cdot}{\wedge} \mu_A(x) = \begin{cases} \mu_A(x) & \text{for } \mu_R(x, y) = 1 \\ 0 & \text{for } \mu_R(x, y) = 0 \end{cases}$$

Using the extensions
$r : \mathcal{P}(X) \to \mathcal{P}(Y), \quad r^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$
of relations R, R^{-1} to **crisp** sets, the extensions to **fuzzy** sets can be written as

$$\mu_{r(A)}(y) = \sup_{x \in r^{-1}(y)} \mu_A(x)$$

$$\mu_{r^{-1}(B)}(x) = \sup_{y \in r(x)} \mu_B(y)$$

If, moreover, R is a mapping, then

$$\mu_{r^{-1}(B)}(x) = \mu_B(r(x))$$

If R^{-1} is a mapping, then

$$\mu_{r(A)}(y) = \mu_A(r^{-1}(y))$$

Theorem

$$r\big(\mathcal{R}_A(\alpha)\big) \subseteq \mathcal{R}_{r(A)}(\alpha)$$

If the sets

$$r^{-1}(y) = \{x \in X : (x, y) \in R\}$$

are finite for all $y \in Y$, then the equality holds.

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Convex fuzzy sets

Here L denotes a linear space.

A crisp set $A \subseteq L$ is called **convex** if

$$\forall x, y \in A\ \forall \lambda \in (0,1) : \lambda x + (1-\lambda)\, y \in A$$

Using membership functions:

$$\min\bigl(\mu_A(x), \mu_A(y)\bigr) \le \mu_A\bigl(\lambda x + (1-\lambda)\, y\bigr)$$

Let X be a crisp convex subset of a linear space.
A fuzzy set $A \in \mathcal{F}(X)$ is called **convex** if

$$\forall x, y \in X\ \forall \lambda \in (0,1) : \quad \mu_A\bigl(\lambda x + (1-\lambda)\, y\bigr) \ge \mu_A(x) \underset{\mathrm{S}}{\wedge} \mu_A(y)$$

Convexity of fuzzy sets has nothing in common with the convexity of its membership function!

Theorem Convexity is cut-consistent.

In particular, a fuzzy set of real numbers is convex iff all its cuts are intervals.

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