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Theorem: [Butnariu, Klement] All elements of $\mathcal{T}$ are $\mathcal{B}$-measurable. Each measure is **regular** and it is of the form

$$\mu(A) = \int A \, d\nu$$

where $\nu = \mu \restriction \mathcal{B}$ is a (classical) measure on $\mathcal{B}$.

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Always: Crisp elements of $\mathcal{T}$, i.e., $\mathcal{T} \cap \{0, 1\}^X$, determine a σ-algebra $\mathcal{B}$

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Full tribes

Example: Let $\mathcal{B}$ be a σ-algebra of subsets of X,
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Łukasiewicz t-norm

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$$

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Theorem: [Butnariu, Klement] All elements of $\mathcal{T}$ are $\mathcal{B}$-measurable. Each measure is **regular** and it is of the form

$$\mu(A) = \int A \, d\nu$$

where $\nu = \mu \restriction \mathcal{B}$ is a (classical) measure on $\mathcal{B}$.

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Basic notions of fuzzy measure theory

classical measure theory	**fuzzy** measure theory
σ-algebra $\mathcal{T} \subseteq \mathbf{2}^X$	**tribe** $(\mathcal{T}, T)$, where $\mathcal{T} \subseteq [\mathbf{0}, \mathbf{1}]^X$
$\emptyset \in \mathcal{T}$	$0 \in \mathcal{T}$
$A \in \mathcal{T} \Rightarrow A' = X \setminus A \in \mathcal{T}$	$A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$
$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$	$A, B \in \mathcal{T} \Rightarrow A \underset{\cdot}{\cap} B \in \mathcal{T}$ *
$(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$	$(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$
measure $\mu\colon \mathcal{T} \to [0, \infty[$	**regular measure** $\mu\colon \mathcal{T} \to [0, \infty[$
$\mu(\emptyset) = 0$	$\mu(0) = 0$
$\mu(A \cup B)$	$\mu(A \dot{\cup} B)$
$= \mu(A) + \mu(B) - \mu(A \cap B)$	$= \mu(A) + \mu(B) - \mu(A \underset{\cdot}{\cap} B)$ *
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