Introduction to Fuzzy Logic

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CLASSICAL LOGIC (CL)

SYNTAX OF CLASSICAL LOGIC 1

$\mathcal{A}$ ... countable set of propositional variables

$\mathcal{L} = \{\rightarrow, \mathbf{0}\}$ ... the set of logical connectives:

$\rightarrow$ ... (binary) implication

$\mathbf{0}$ ... (nulary) false

Formulas

- all elements of $\mathcal{A}$ are formulas
- $\mathbf{0}$ is a formula
- if A, B are formulas, then $A \rightarrow B$ is a formula

More exactly, we use brackets like $(A) \rightarrow (B)$

Derived connectives:

$\neg A = A \to \mathbf{0}$ ... (unary) negation

$\mathbf{1} = \neg\mathbf{0} = \mathbf{0} \to \mathbf{0}$ ... (nulary) true

$A \wedge B = \neg(A \to \neg B)$ ... (binary) conjunction

$A \vee B = \neg A \to B$ ... (binary) disjunction

$A \leftrightarrow B = (A \to B) \wedge (B \to A)$ ... (binary) equivalence

SEMANTICS OF CLASSICAL LOGIC

In general: a Boolean algebra, it is enough to consider

Standard semantics

$\{0, 1\}$ ... the set of truth values

$\to$ ... (interpreted as) Boolean implication $\Rightarrow$

$\mathbf{0}$ ... (interpreted as) 0

Interpretation of derived connectives:

$\neg$ ... Boolean negation

$\mathbf{1}$ ... 1

$\wedge$ ... conjunction

$\vee$ ... disjunction

$\leftrightarrow$ ... Boolean equivalence $\Leftrightarrow$

An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Tautology is a formula A which is *always* evaluated to 1

Notation: $\models A$

Moreover, for any set of formulas $\mathcal{T}$,

$\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T} : e(B) = 1$.

Contradiction is a formula which is *always* evaluated to 0

A formula is **satisfiable** if it is evaluated to 1 for *at least one evaluation*

SYNTAX OF CLASSICAL LOGIC 2

Logical axioms

(C1) $A \to (B \to A)$

(C2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(C3) $(\neg A \to \neg B) \to (B \to A)$

Deduction rule: Modus Ponens

MP$(A,\ A \to B)$: $\dfrac{A,\ A \to B}{B}$

Theory $\mathcal{T}$ ... set of formulas (**special axioms**)

Provable formula (=**theorem**) in theory $\mathcal{T}$ is a formula which admits a **proof**, i.e., a finite sequence of formulas such that each of them is

- a special axiom (=element of $\mathcal{T}$), or
- an instance of a logical axiom (obtained by a substitution), or
- a result of application of a deduction rule to preceding formulas in the proof.

Notation:

$\mathcal{T} \vdash A$

$B \vdash A$ (for $\mathcal{T} = \{B\}$)

$\vdash A$ (for $\mathcal{T} = \emptyset$)

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Example CI1 $A \vdash B \rightarrow A$

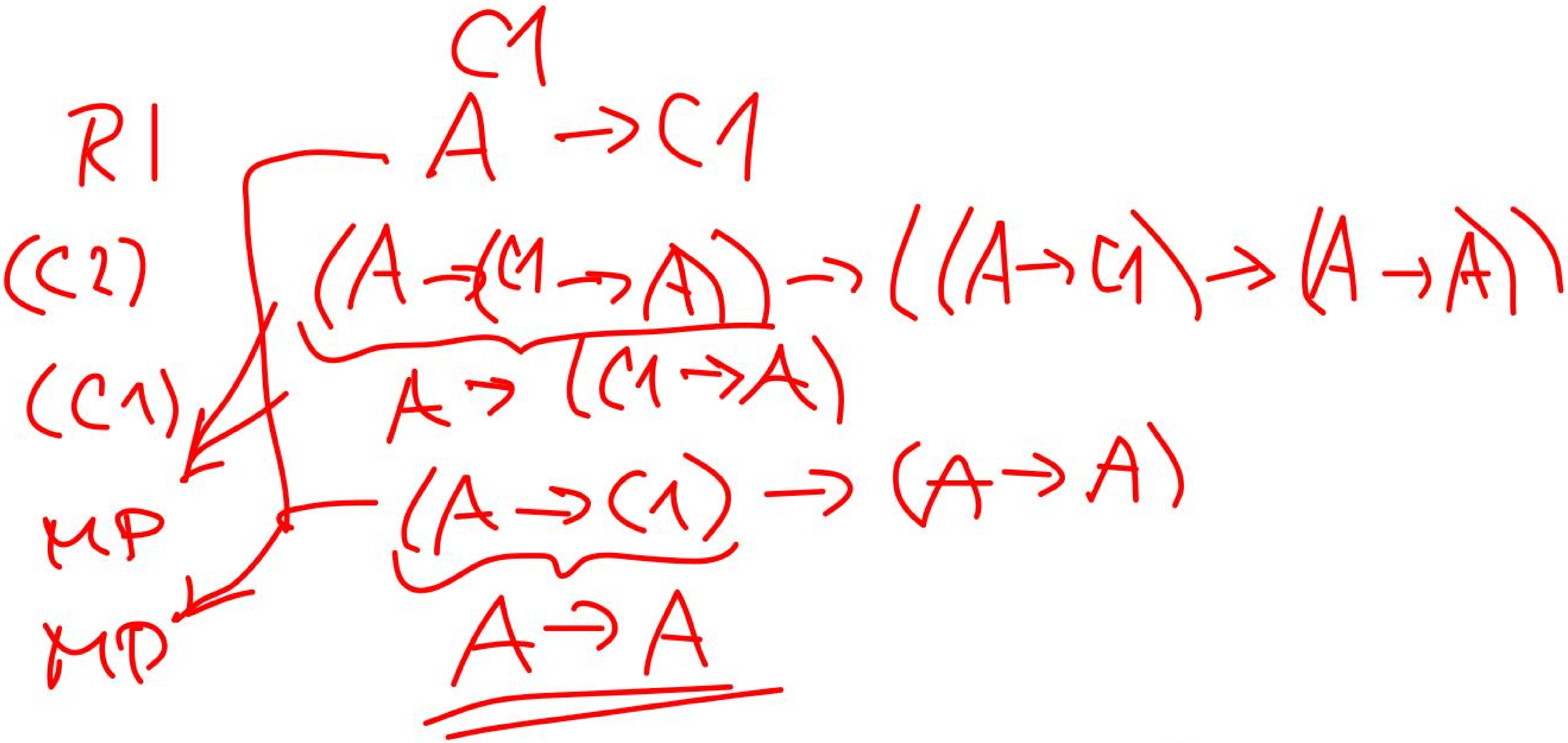
Example CI1 $\quad A \vdash B \to A$

$$\begin{aligned} \text{(C1)}: &\quad D_1 = A \to (B \to A) \\ \text{SA (special axiom)}: &\quad D_2 = A \\ \text{MP}(D_2, D_1): &\quad D_3 = B \to A \end{aligned}$$

$\Rightarrow$ we can add a deduction rule RI(A): $\dfrac{A}{B \to A}$

Example CI2 $\vdash A \rightarrow A$

Example CI2 $\vdash A \to A$



Example Cl2 $\vdash A \to A$

For brevity, let B denote any provable formula, e.g., axiom (C1).

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\text{RI}(D_1): &\quad D_2 = A \to B \\
\text{(C2)}, C := A: &\quad D_3 = (A \to (B \to A)) \to ((A \to B) \to (A \to A)) \\
\text{(C1)}: &\quad D_4 = A \to (B \to A) \\
\text{MP}(D_4, D_3): &\quad D_5 = (A \to B) \to (A \to A) \\
\text{MP}(D_2, D_5): &\quad D_6 = A \to A
\end{aligned}$$

$\Rightarrow$ we can add an axiom (AA): $A \to A$

Corollary Cor1 $\vdash \mathbf{0} \to \mathbf{0}$, $\vdash \neg\mathbf{0}$, $\vdash \mathbf{1}$

Example CI3 $\vdash A \to \mathbf{1}$ for all A

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$$\frac{1}{A \to 1} \text{ R1}$$

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Example CI3 $\vdash A \to \mathbf{1}$ for all A

$$\mathbf{1}$$

R1 $\quad A \to 1$

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Example Cl3 $\vdash A \to \mathbf{1}$ for all A

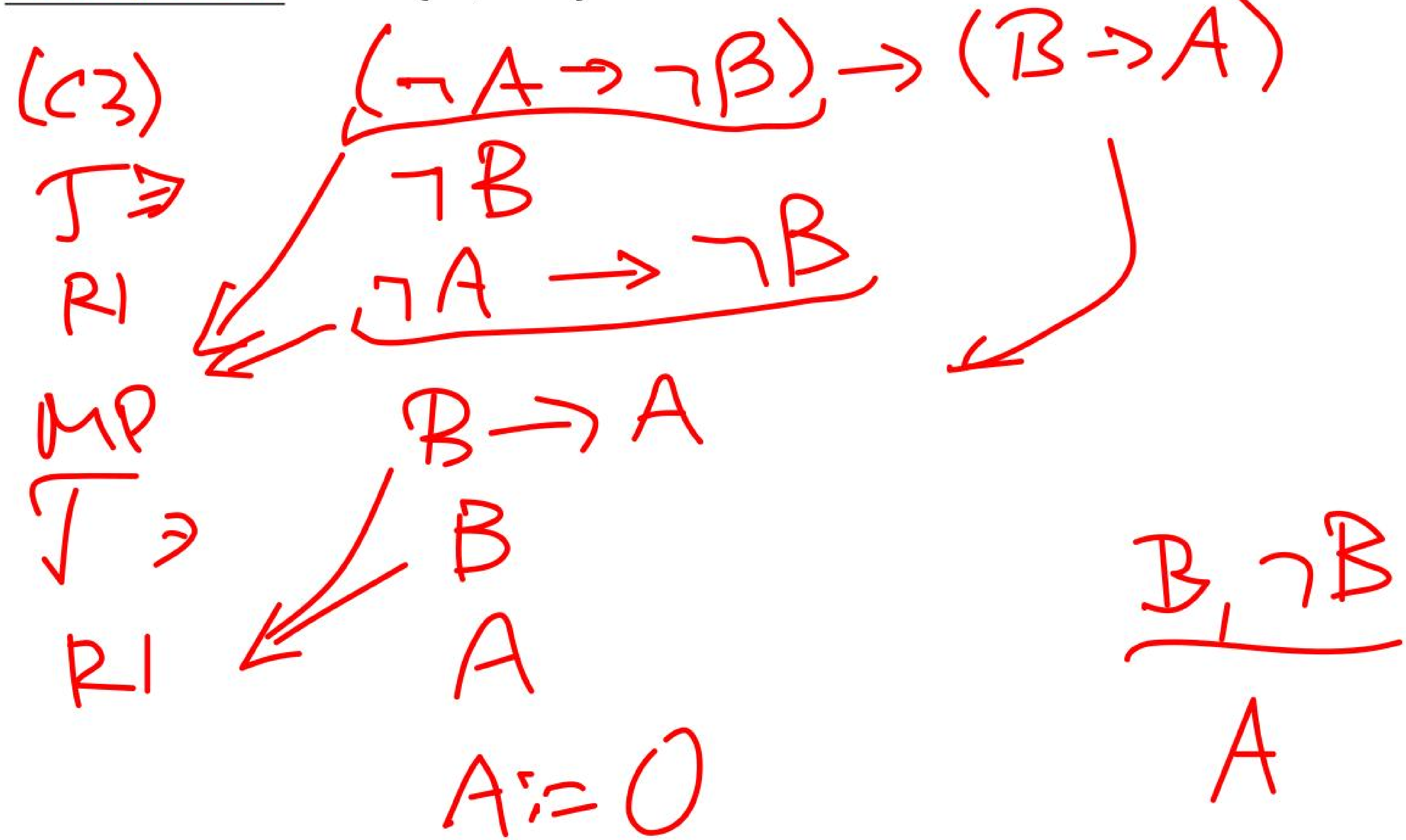
$$\begin{aligned} \text{Cor1}: &\quad D_1 = \mathbf{1} \\ \text{RI}(D_1): &\quad D_2 = A \to \mathbf{1} \end{aligned}$$

Example Cl4 $\{B, \neg B\} \vdash A$ for all A

Example Cl3 $\vdash A \to \mathbf{1}$ for all A

$$\begin{aligned} \text{Cor1}: \quad & D_1 = \mathbf{1} \\ \text{RI}(D_1): \quad & D_2 = A \to \mathbf{1} \end{aligned}$$

Example Cl4 $\{B, \neg B\} \vdash A$ for all A



Example Cl3 $\vdash A \to \mathbf{1}$ for all A

$$\begin{aligned} \text{Cor1}: \quad & D_1 = \mathbf{1} \\ \text{RI}(D_1): \quad & D_2 = A \to \mathbf{1} \end{aligned}$$

Example Cl4 $\{B, \neg B\} \vdash A$ for all A

$$\begin{aligned} \text{(C3)}: \quad & D_1 = (\neg A \to \neg B) \to (B \to A) \\ \text{SA}: \quad & D_2 = \neg B \\ \text{RI}(D_2): \quad & D_3 = \neg A \to \neg B \\ \text{MP}(D_3, D_1): \quad & D_4 = B \to A \\ \text{SA}: \quad & D_5 = B \\ \text{MP}(D_5, D_4): \quad & D_6 = A \end{aligned}$$

$\Rightarrow$ we can add a deduction rule ALL(B) : $\dfrac{B, \neg B}{A}$

Example CI5 $\{A \to B, B \to C\} \vdash A \to C$

Example CI5

$\{A \to B, B \to C\} \vdash A \to C$

$J \ni \quad A \to B$

$J \ni \quad B \to C$

R1 $\quad A \to (B \to C)$

(C2) $\quad (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

MP $\quad (A \to B) \to (A \to C)$

MP $\quad A \to C$

Example CI5 $\{A \to B, B \to C\} \vdash A \to C$

$$
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\text{RI}(D2): \quad & D_3 = A \to (B \to C) \\
\text{(C2)}: \quad & D_4 = (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\
\text{MP}(D_3, D_4): \quad & D_5 = (A \to B) \to (A \to C) \\
\text{MP}(D_1, D_5): \quad & D_6 = A \to C
\end{aligned}
$$

$\Rightarrow$ we can add a deduction rule TI($A \to B, B \to C$): $\dfrac{A \to B, B \to C}{A \to C}$

(transitivity of implication)

Example CI5 $\{A \to B, B \to C\} \vdash A \to C$

$$\begin{aligned}
\text{(SA1)}: &\quad D_1 = A \to B \\
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\text{RI}(D2): &\quad D_3 = A \to (B \to C) \\
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$\Rightarrow$ we can add a deduction rule TI($A \to B, B \to C$): $\dfrac{A \to B, B \to C}{A \to C}$

(transitivity of implication)

Example Cl6 $\{E \to T, E \to \neg T\} \vdash \neg E$

Example CI6 $\{E \to T, E \to \neg T\} \vdash \neg E$

$E \to T$

$E \to \neg T = E \to (T \to 0)$

IC2 $(E \to (T \to 0)) \to ((E \to T) \to (E \to 0))$

MP $(E \to T) \to (E \to 0)$

MP $E \to 0 = \neg E$

Example CI6 $\{E \to T, E \to \neg T\} \vdash \neg E$

$$
\begin{aligned}
\text{(SA1)}: \quad & D_1 = E \to T \\
\text{(SA2)}: \quad & D_2 = E \to \neg T = E \to (T \to \mathbf{0}) \\
\text{(C2)}: \quad & D_3 = (E \to (T \to \mathbf{0})) \to ((E \to T) \to (E \to \mathbf{0})) \\
\text{MP}(D_2, D_3): \quad & D_4 = (E \to T) \to (E \to \mathbf{0}) \\
\text{MP}(D_1, D_4): \quad & D_5 = E \to \mathbf{0} = \neg E
\end{aligned}
$$

Example CI7 $\vdash \mathbf{0} \to A$ for all A (*ex falso quodlibet*)

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(C3) $(\neg A \to \neg 0) \to (0 \to A)$

$\neg A \to \neg 0$

MP

$0 \to A$

Example CI7 $\vdash \mathbf{0} \to A$ for all A (*ex falso quodlibet*)

$$\begin{aligned} \text{(C3) } B := \mathbf{0} : &\quad D_1 = (\neg A \to \neg \mathbf{0}) \to (\mathbf{0} \to A) \\ \text{CI3, } A := \neg A : &\quad D_2 = \neg A \to \neg \mathbf{0} \\ \text{MP}(D_2, D_1) : &\quad D_3 = \mathbf{0} \to A \end{aligned}$$

$\Rightarrow$ we can add an axiom $\mathbf{0} \to A$

Example CI8 $\vdash A \vee \neg A$ for all A (*tertium non datur*)

Example Cl7 $\vdash \mathbf{0} \to A$ for all A (*ex falso quodlibet*)

$$\begin{aligned} \text{(C3) } B := \mathbf{0}: \quad & D_1 = (\neg A \to \neg \mathbf{0}) \to (\mathbf{0} \to A) \\ \text{Cl3, } A := \neg A: \quad & D_2 = \neg A \to \neg \mathbf{0} \\ \text{MP}(D_2, D_1): \quad & D_3 = \mathbf{0} \to A \end{aligned}$$

$\Rightarrow$ we can add an axiom $\mathbf{0} \to A$

Example Cl8 $\vdash A \vee \neg A$ for all A (*tertium non datur*)

$$\neg A \to \neg A$$

$$B \to B$$

Example Cl7 $\vdash \mathbf{0} \to A$ for all A (*ex falso quodlibet*)

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$\Rightarrow$ we can add an axiom $\mathbf{0} \to A$

Example Cl8 $\vdash A \vee \neg A$ for all A (*tertium non datur*)

$$\text{Cl2, } A := \neg A : \quad D_1 = \neg A \to \neg A = A \vee \neg A$$

$\Rightarrow$ we can add an axiom $A \vee \neg A$ (in contrast to $\neg A \vee A$ which is also provable, but not yet proved)

Example CI9 $B \vdash A \vee B$ for all A, B

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B

R1 $\neg A \to B = A \vee B$