Deduction theorem in classical logic

$\mathcal{T}$ ... theory

A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \to B$

Proof $\Leftarrow$:

	//BEGIN of proof of $\mathcal{T} \vdash A \to B$
	$\vdots$
	$D_{i-1} = A \to B$
	//END of proof of $\mathcal{T} \vdash A \to B$
SA :	$D_i = A$
MP(D_i, D_{i-1}) :	$D_{i+1} = B$

$\Rightarrow$: Proof by contradiction:

Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \to B$.

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Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \to B$.

1. B is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

$$\begin{aligned} & D_1 = B \\ RI(D_1): \quad & D_2 = A \to B \end{aligned}$$

hence $\mathcal{T} \vdash A \to B$.

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Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \to B$.

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hence $\mathcal{T} \vdash A \to B$.

2. $B \neq A$ because $\mathcal{T} \vdash A \to A$.

3. B is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.

WLOG, we choose for B a formula with the shortest possible proof; its shortest proof must be of the following form:

$$
\begin{array}{rl}
 & \vdots \\
 & D_i \\
 & \vdots \\
 & D_j = D_i \to B \\
 & \vdots \\
\text{MP}(D_i, D_j): & D_m = B
\end{array}
$$

for $i < j < m$ or $j < i < m$.

3. B is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.
WLOG, we choose for B a formula with the shortest possible proof; its shortest proof must be of the following form:

$$\begin{array}{rl} & \vdots \\ & D_i \\ & \vdots \\ & D_j = D_i \to B \\ & \vdots \\ \text{MP}(D_i, D_j): & D_m = B \end{array}$$

for $i < j < m$ or $j < i < m$.

The proofs of $\mathcal{T} \cup \{A\} \vdash D_i$, $\mathcal{T} \cup \{A\} \vdash D_j$ are of lengths $< m$, therefore

$$\mathcal{T} \vdash A \to D_i$$
$$\mathcal{T} \vdash A \to D_j = A \to (D_i \to B)$$

Proof of $\mathcal{T} \vdash A \to B$:

	//BEGIN of proof of $\mathcal{T} \vdash A \to D_i$
	$\vdots$
	$D_k = A \to D_i$
	//END of proof of $\mathcal{T} \vdash A \to D_i$
	//BEGIN of proof of $\mathcal{T} \vdash A \to D_j$
	$\vdots$
	$D_n = A \to \overbrace{(D_i \to B)}^{D_j}$
	//END of proof of $\mathcal{T} \vdash A \to D_j$
(C2) $B := D_i, C := B$:	$D_{n+1} = (A \to (D_i \to B)) \to ((A \to D_i) \to (A \to B))$
MP(D_n, D_{n+1}) :	$D_{n+2} = (A \to D_i) \to (A \to B)$
MP(D_k, D_{n+2}) :	$D_{n+3} = A \to B$

Proof of $\mathcal{T} \vdash A \to B$:

//BEGIN of proof of $\mathcal{T} \vdash A \to D_i$

$\vdots$

$D_k = \boxed{A \to D_i}$

//END of proof of $\mathcal{T} \vdash A \to D_i$

//BEGIN of proof of $\mathcal{T} \vdash A \to D_j$

$\vdots$

$D_n = \boxed{A \to \overbrace{(D_i \to B)}^{D_j}}$

//END of proof of $\mathcal{T} \vdash A \to D_j$

(C2) $B := D_i, C := B$: $D_{n+1} = \boxed{(A \to (D_i \to B))} \to \boxed{((A \to D_i) \to (A \to B))}$

MP(D_n, D_{n+1}) : $D_{n+2} = \boxed{(A \to D_i)} \to (A \to B)$

MP(D_k, D_{n+2}) : $D_{n+3} = \boxed{A \to B}$

Corollary Cor2 $A \vdash A \vee B$ for all A, B

$$A \vdash \neg A \to B = A \vee B$$

$$\Updownarrow \text{(DT)}$$

$$\text{ALL}(A): \quad \{A, \neg A\} \vdash B$$

$\Rightarrow$ we can add a deduction rule $\dfrac{A}{A \vee B}$ (and $\dfrac{B}{A \vee B}$ was already proved in Cl9)

Corollary Cor2

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$\Rightarrow$ we can add a deduction rule $\dfrac{A}{A \vee B}$ (and $\dfrac{B}{A \vee B}$ was already proved in Cl9)

Corollary Cor3 $A \vdash \neg\neg A$, $\vdash A \to \neg\neg A$ for all A

$$\vdash A \to \overbrace{(\neg A \to \mathbf{0})}^{\neg\neg A}$$

$$\Updownarrow \text{(DT)}$$

$$A \vdash \neg A \to \mathbf{0}$$

$$\Updownarrow \text{(DT)}$$

$$\text{ALL}(A): \quad \{A, \neg A\} \vdash \mathbf{0}$$

Corollary Cor4 $\neg\neg A \vdash A$, $\vdash \neg\neg A \to A$ for all A

Cor3, $A := \neg A$: $D_1 = \neg A \to \neg\neg\neg A$

(C3) $B := \neg\neg A$: $D_2 := (\neg A \to \neg\neg\neg A) \to (\neg\neg A \to A)$

MP(D_1, D_2) : $D_3 = \neg\neg A \to A$

Corollary Cor5 $\vdash A \leftrightarrow \neg\neg A$ (can be added to axioms)

How can we simplify our proofs?

$B \leftrightarrow C \vdash (A \to B) \leftrightarrow (A \to C)$

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$$A \vdash \neg A \to \mathbf{0}$$

$$\Updownarrow \text{(DT)}$$

$$\text{ALL}(A): \quad \{A, \neg A\} \vdash \mathbf{0}$$

Corollary Cor4 $\neg\neg A \vdash A$, $\vdash \neg\neg A \to A$ for all A

$$\text{Cor3}, A := \neg A: \quad D_1 = \boxed{\neg A \to \neg\neg\neg A}$$

$$\text{(C3)}\ B := \neg\neg A: \quad D_2 := \boxed{(\neg A \to \neg\neg\neg A)} \to (\neg\neg A \to A)$$

$$\text{MP}(D_1, D_2): \quad D_3 = \neg\neg A \to A$$

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