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$$(\alpha \wedge_S \beta) \vee_S (\alpha \wedge_S \neg_S \beta) \stackrel{?}{=} \alpha$$

As1: $\alpha \le \beta$

$\underbrace{\quad}_{\alpha} \vee_S \; ?$

As2: $\alpha > \beta$

As3: $\alpha \le \neg_S \beta$, $\underbrace{\beta \vee_S \alpha}_{\beta} \stackrel{?}{=} \alpha$

$\neq$

$\alpha = \frac{3}{4}$ $\quad \beta = \frac{1}{2}$

$$\underbrace{\underbrace{\left(\tfrac{3}{4} \wedge_S \tfrac{1}{2}\right)}_{\frac{1}{2}} \vee_S \underbrace{\left(\tfrac{3}{4} \wedge_S \tfrac{1}{2}\right)}_{\frac{1}{2}}}_{\frac{1}{2}} \neq \frac{3}{4}$$

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$$(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \stackrel{?}{=} \alpha$$

P L P

$$\alpha \cdot \beta + \alpha(1-\beta) = \alpha(\beta + 1 - \beta) = \alpha \leq 1$$

$$< 1$$

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Fuzzy propositional algebras

equations written in **black** always hold
equations written in **red** hold for the standard fuzzy operations, but not for some others
equations written in **blue** hold only for some choices of fuzzy operations (not for the standard ones)

$$
\begin{array}{rclrcl}
\underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\alpha & = & \alpha, & & & \\
\alpha \mathbin{\dot{\vee}} \beta & = & \beta \mathbin{\dot{\vee}} \alpha, & \alpha \mathbin{\underset{\cdot}{\wedge}} \beta & = & \beta \mathbin{\underset{\cdot}{\wedge}} \alpha, \\
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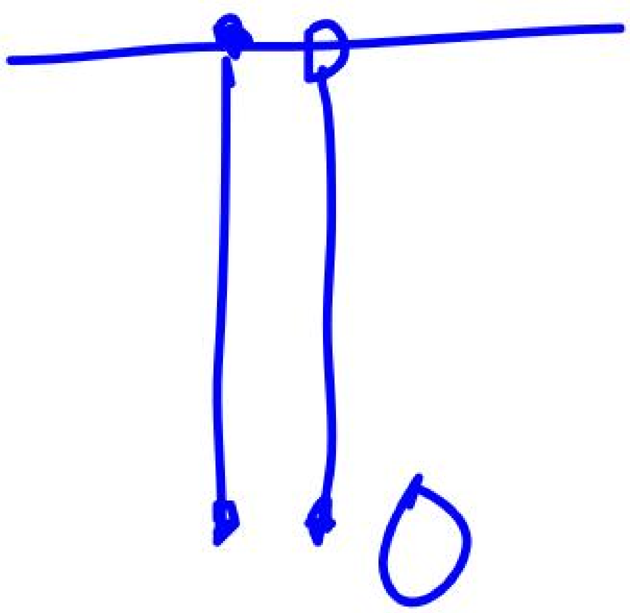

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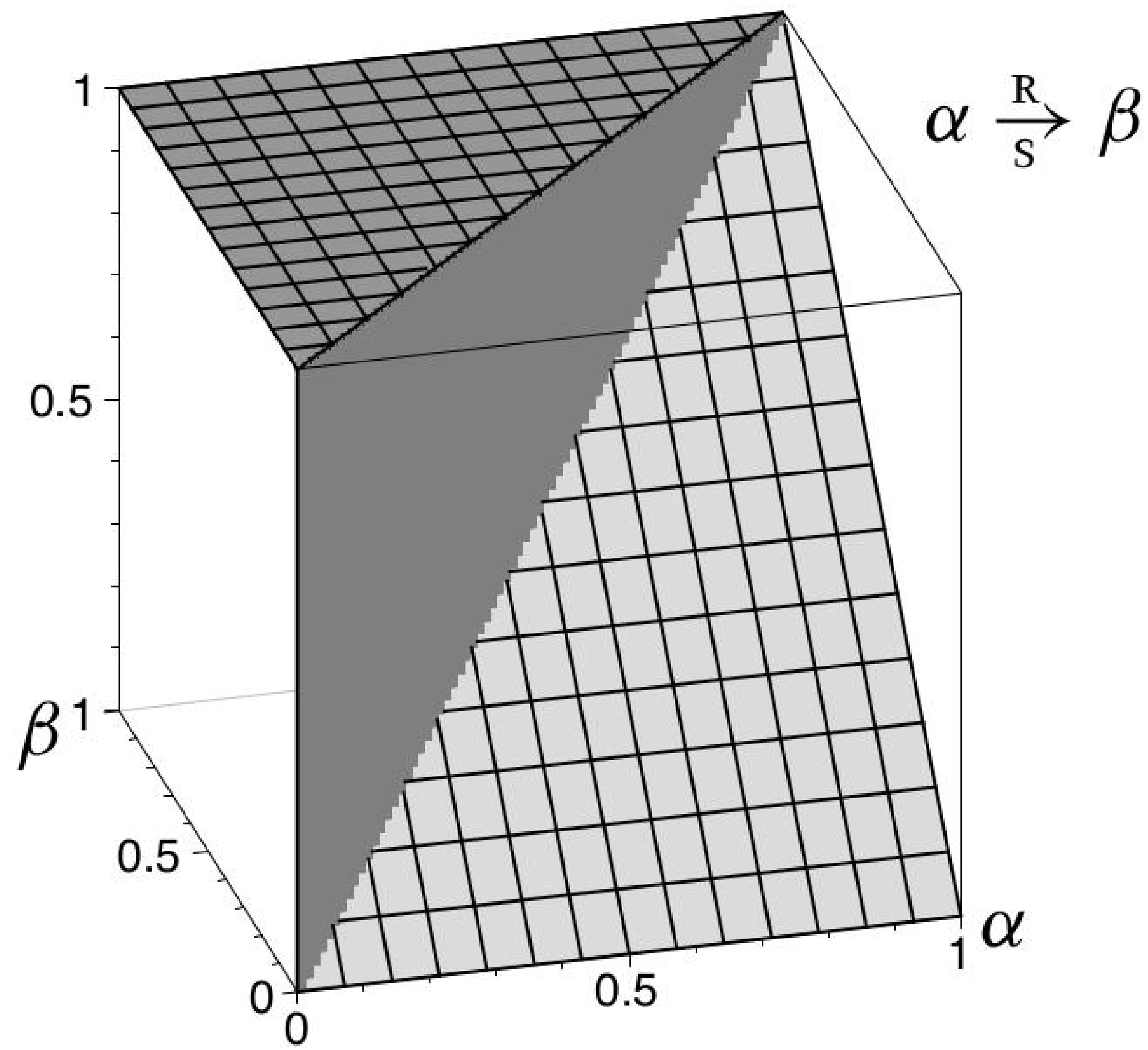

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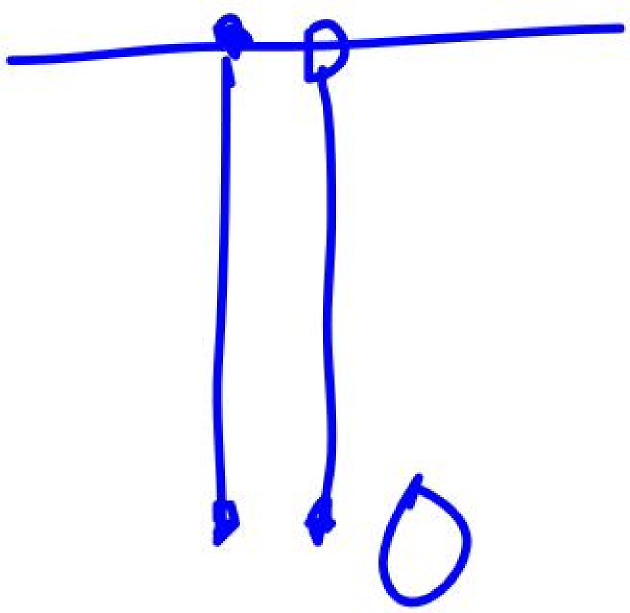

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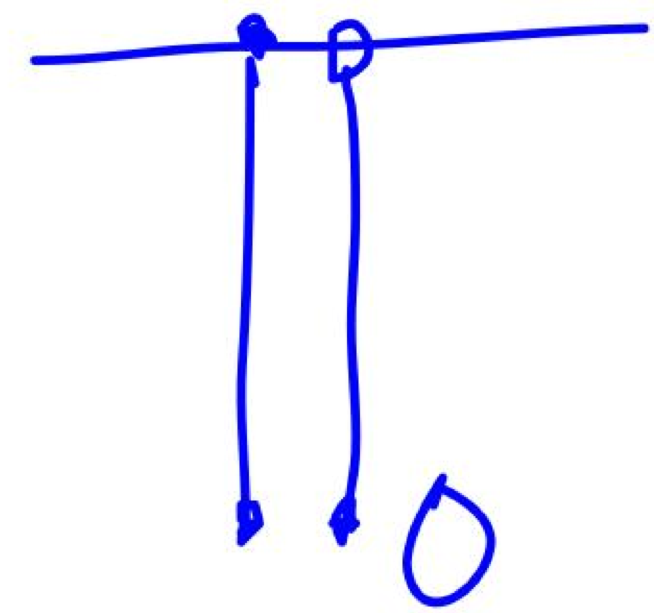

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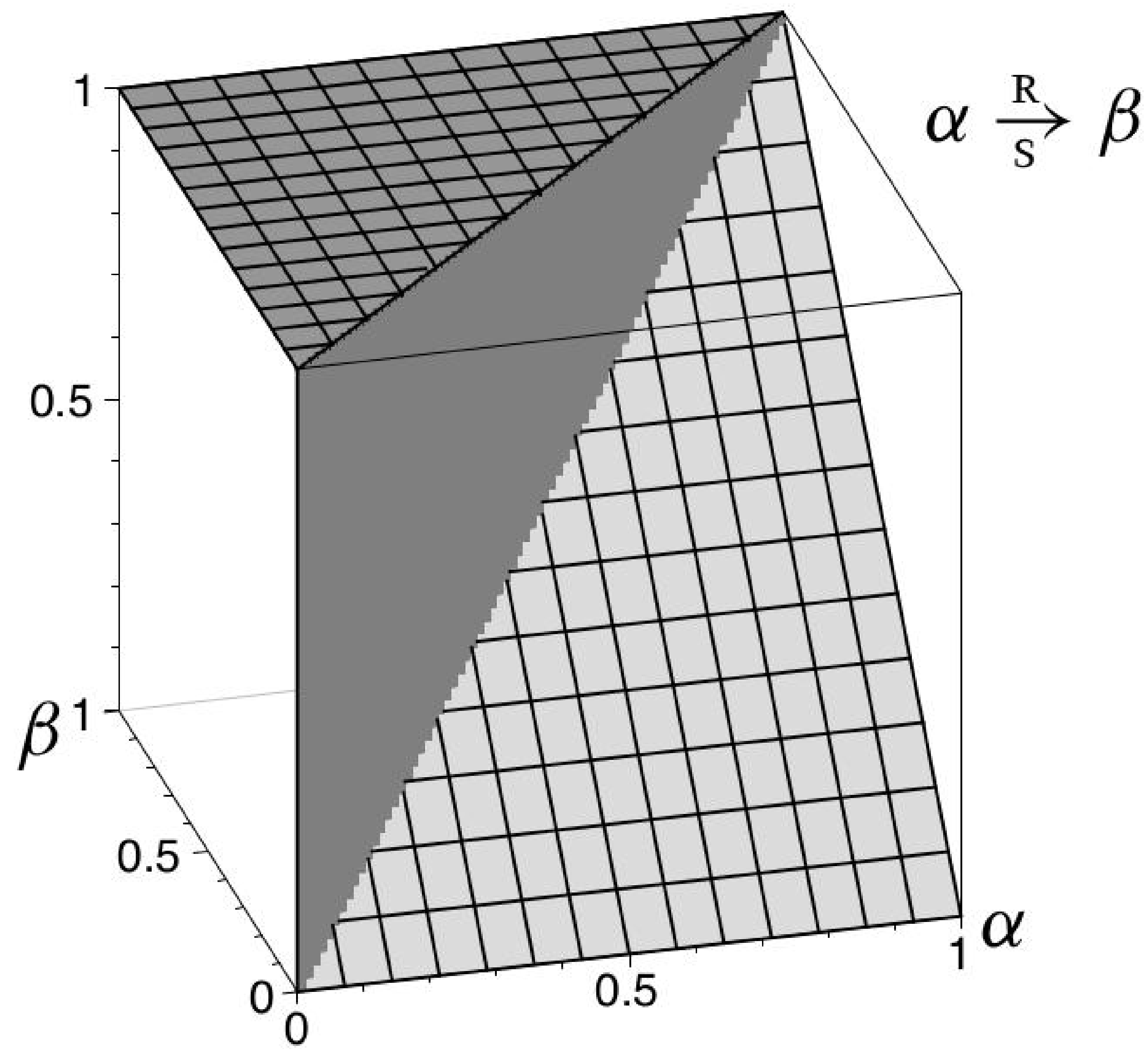

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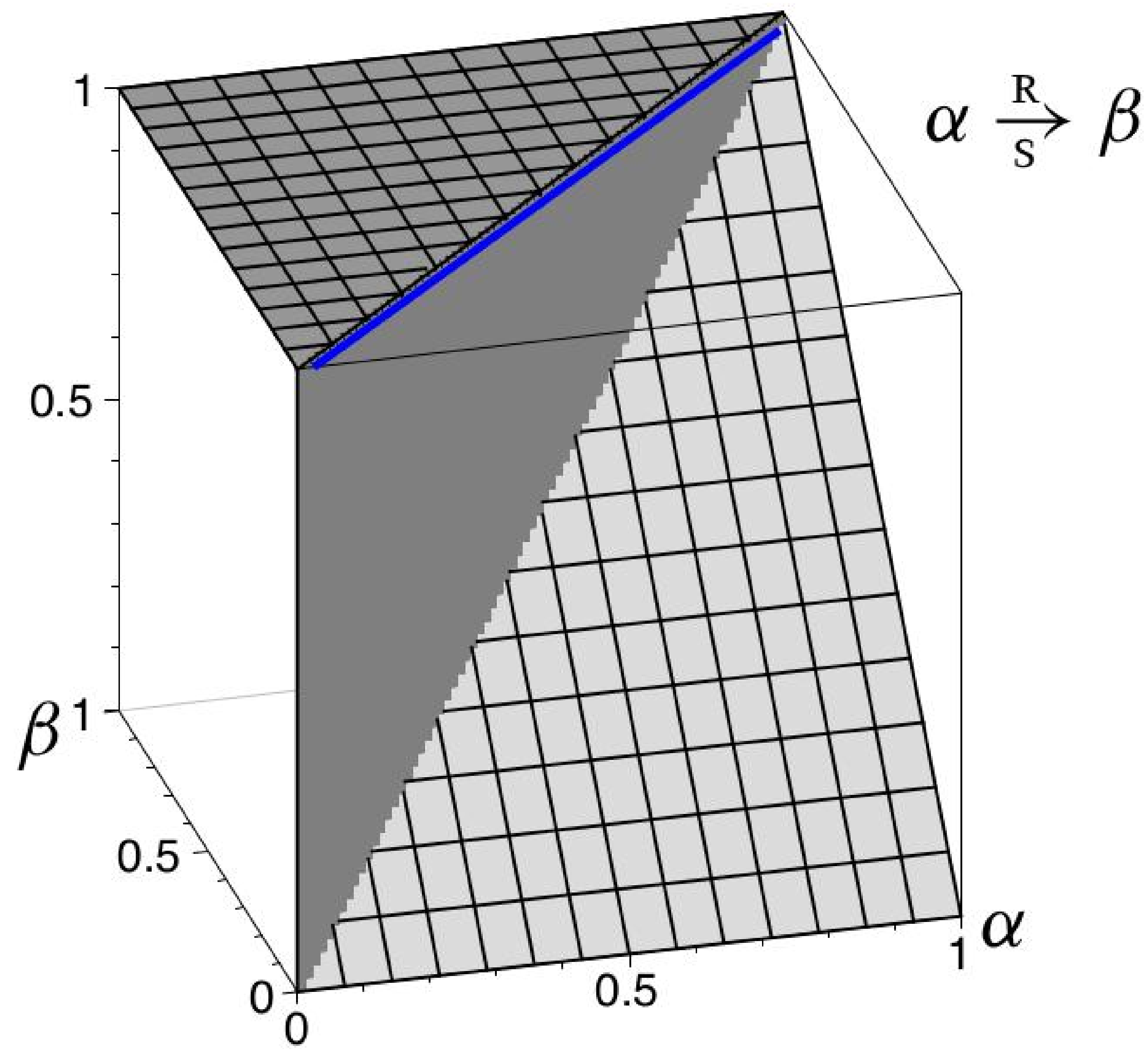

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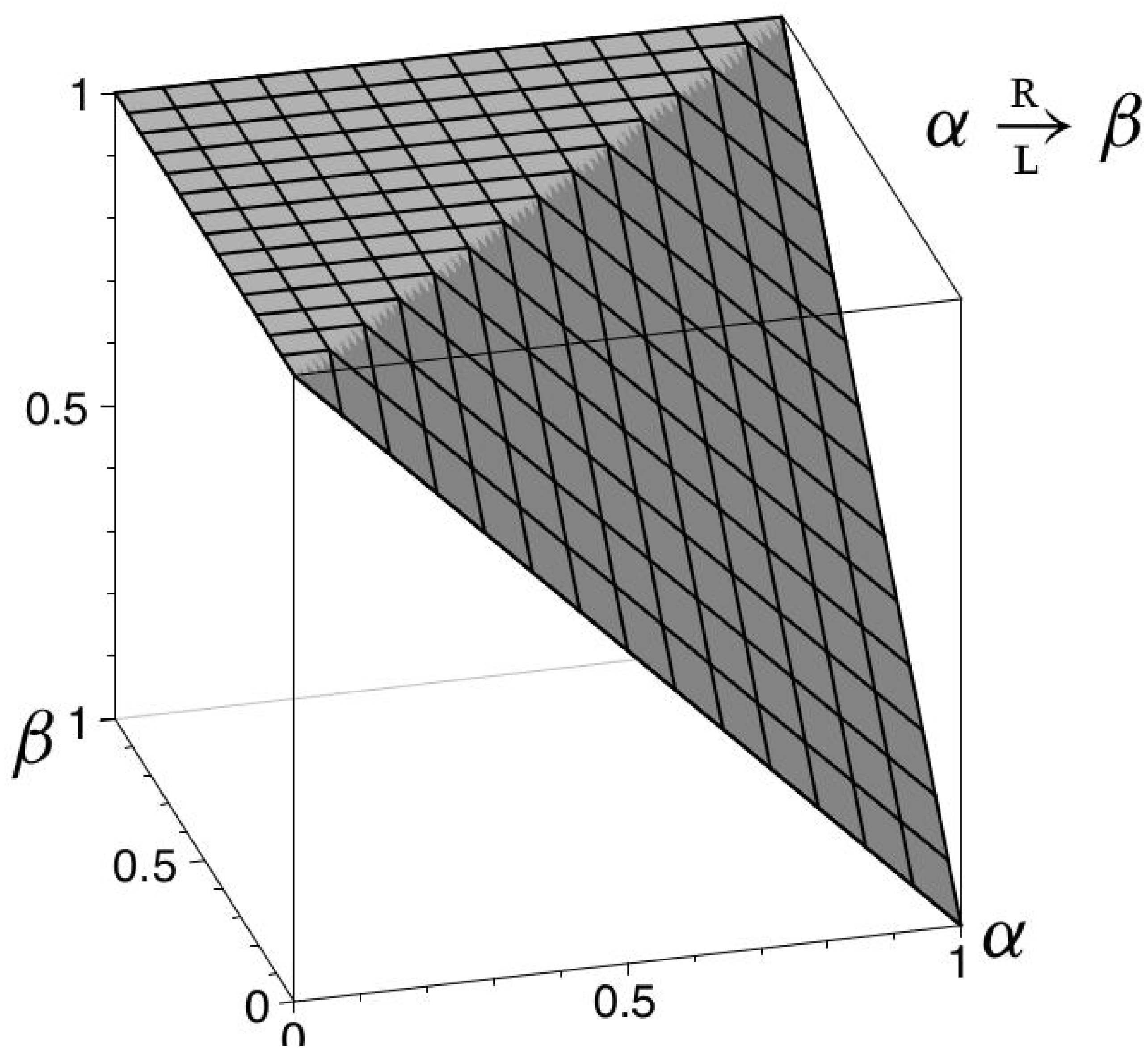

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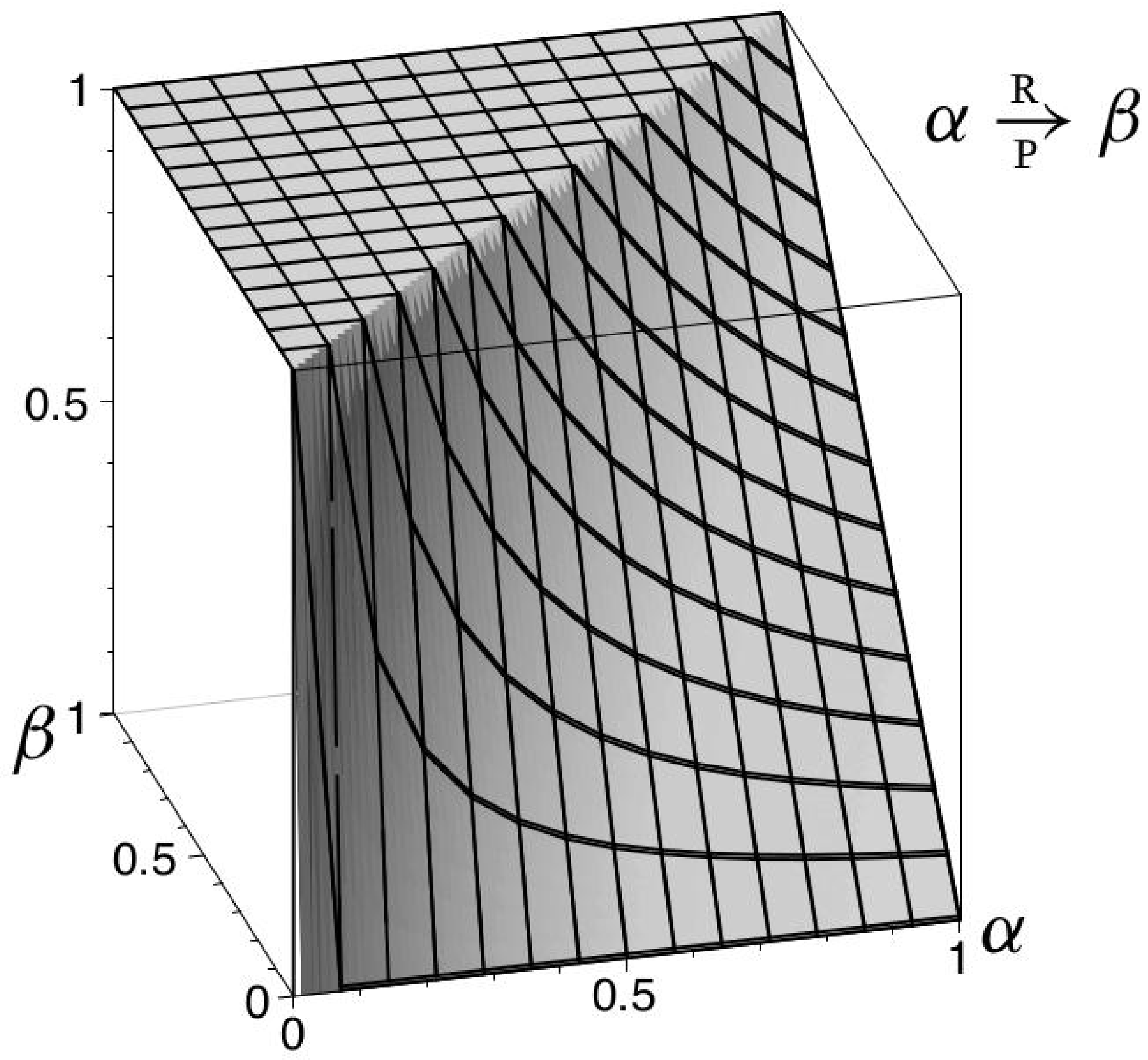

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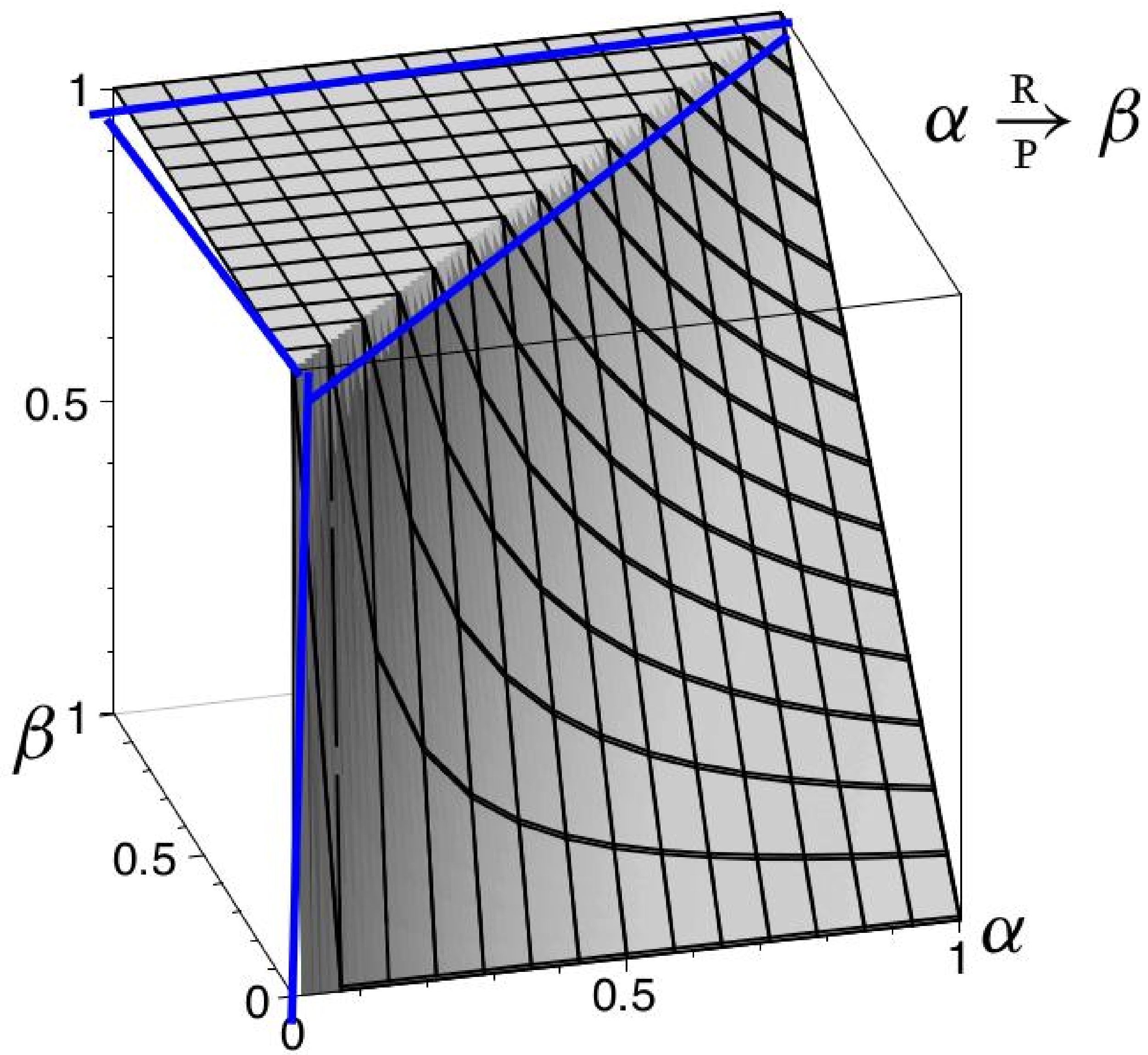

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Theorem: Let $\dot{\wedge}$ be a continuous fuzzy conjunction. Then the R-implication $\overset{\mathrm{R}}{\dot{\to}}$ satisfies (I1a), (I1b), (I2), (I3).

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(I3): When α increases, $\Gamma(\alpha, \beta)$ does not increase.
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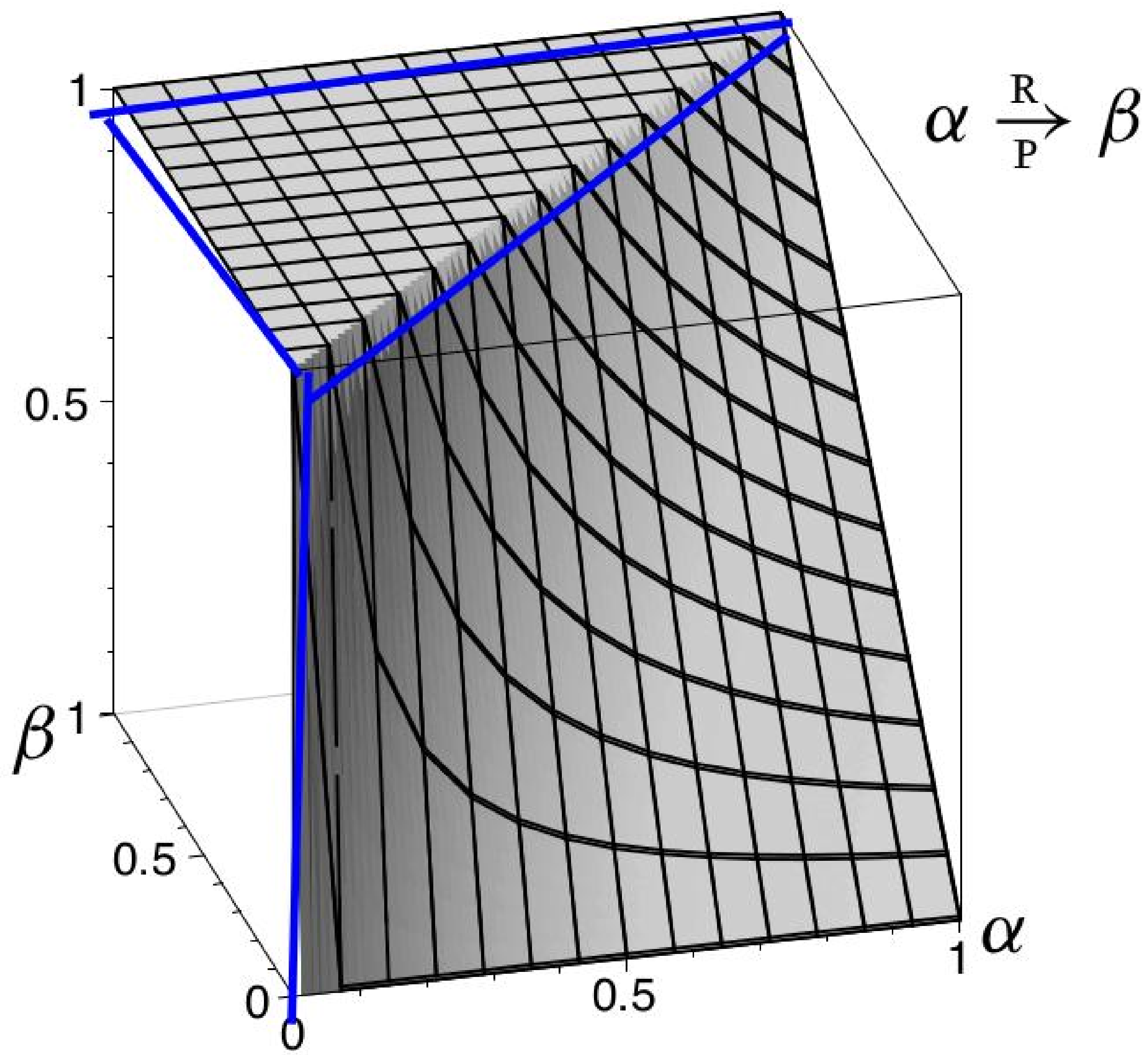

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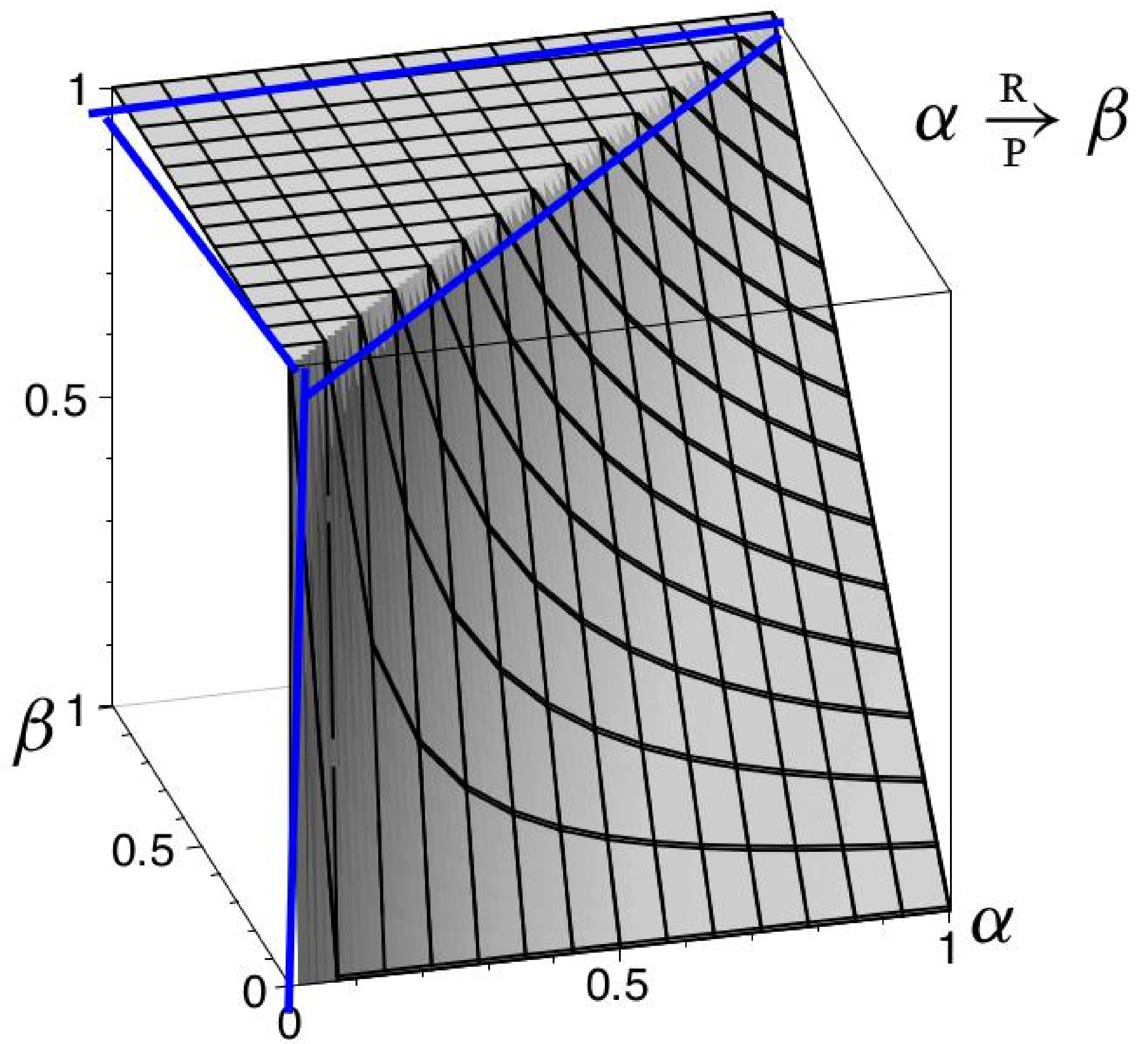

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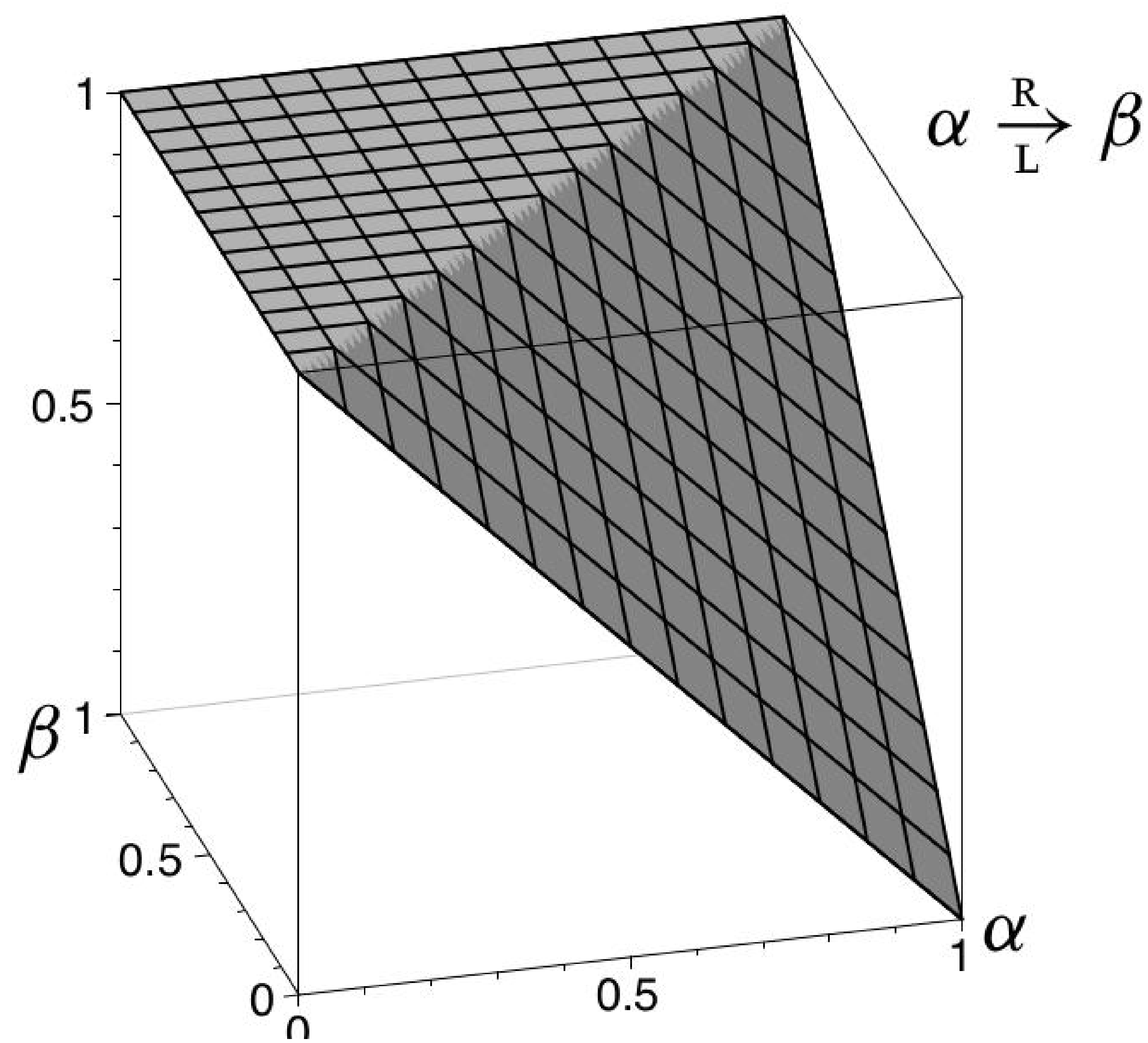

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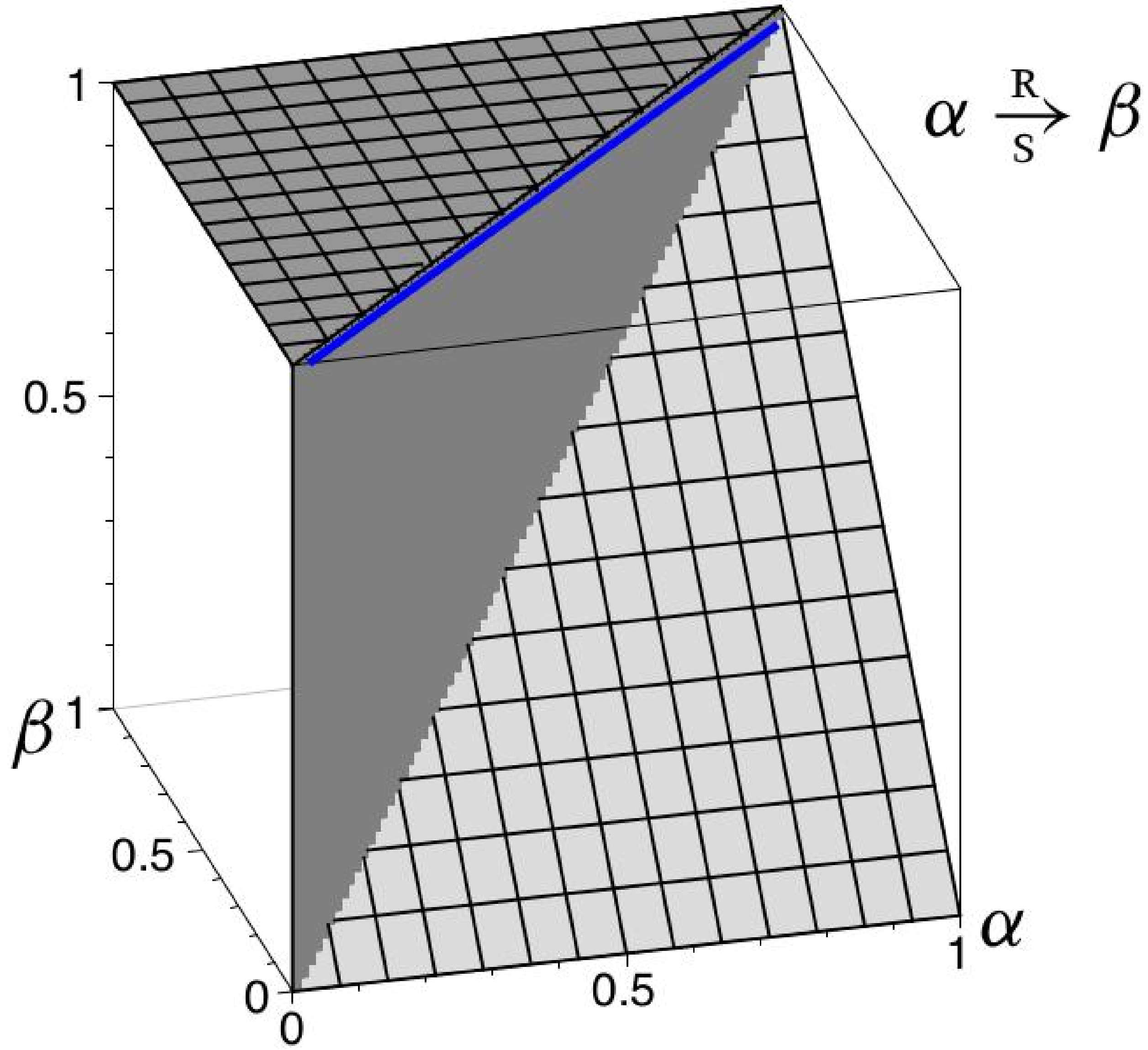

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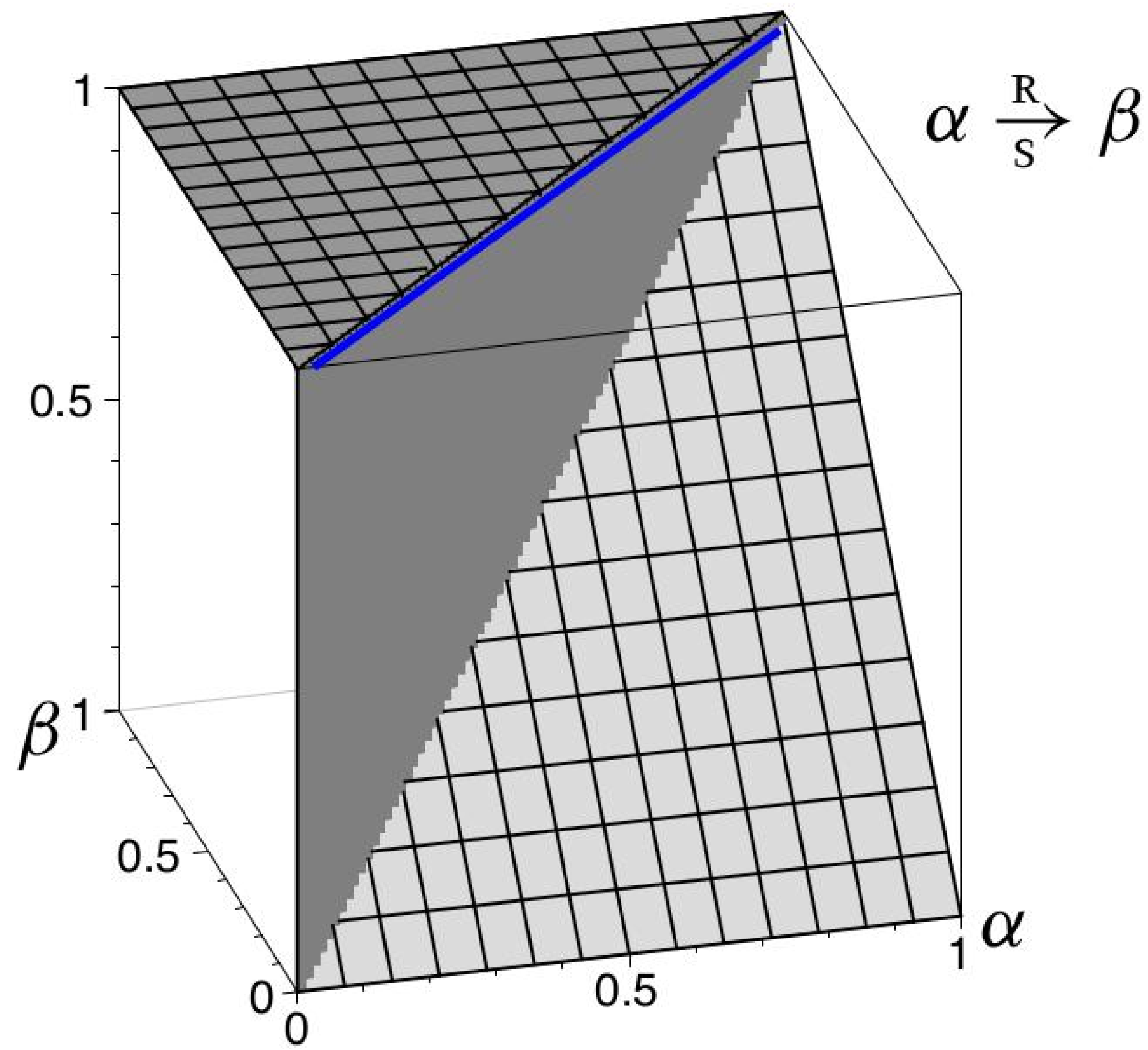

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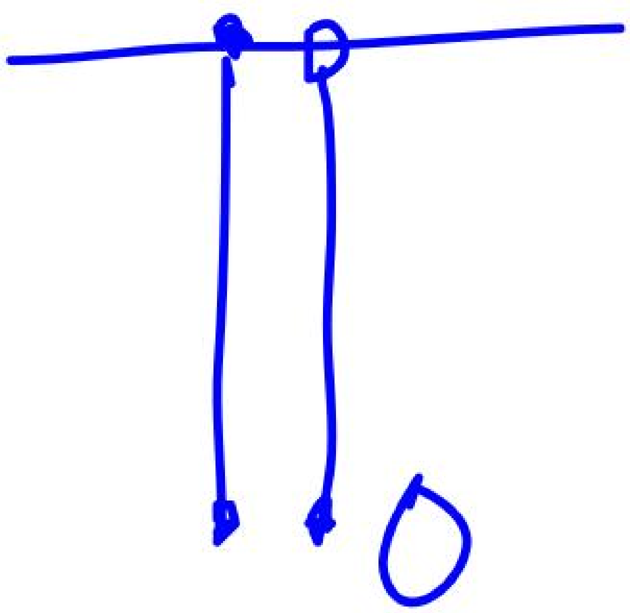

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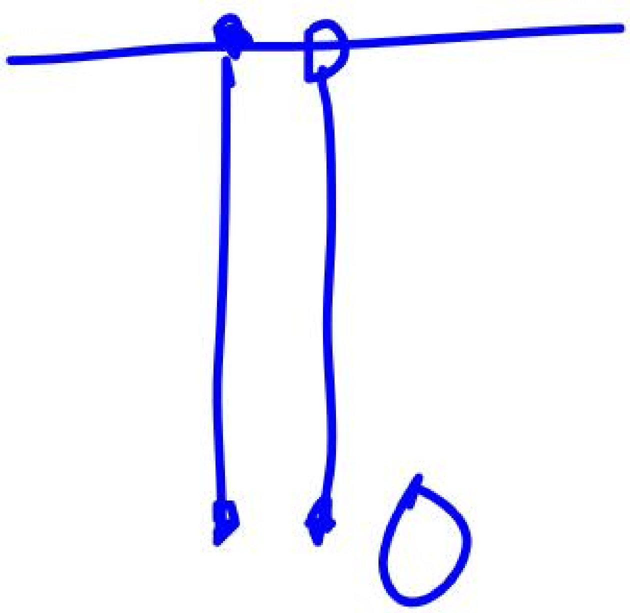

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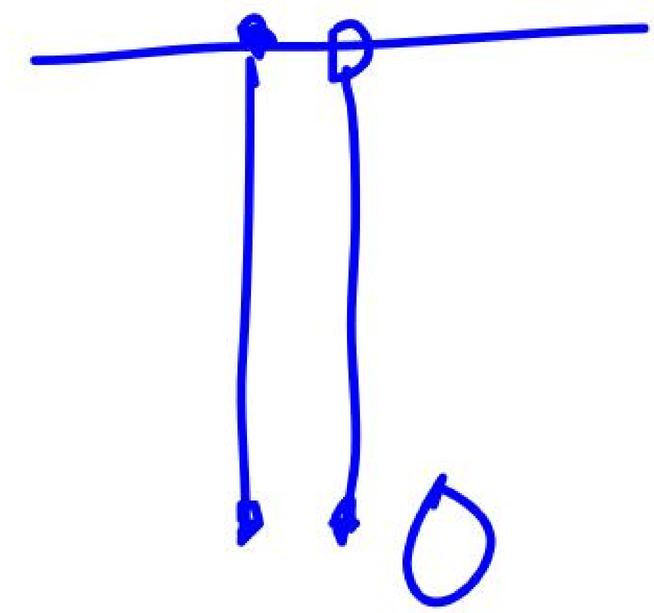

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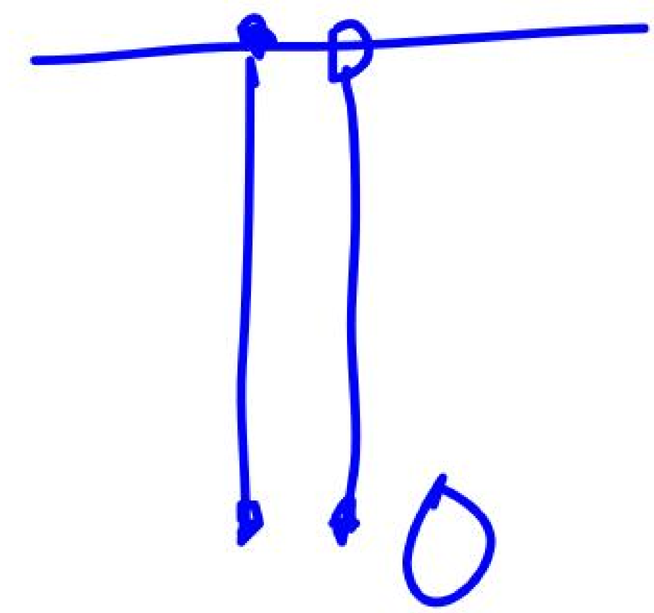

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$$\alpha \overset{\cdot}{\underset{\cdot}{\to}} \beta = 1 \Rightarrow \alpha \le \beta\,, \tag{I1b}$$

$$1 \overset{\cdot}{\underset{\cdot}{\to}} \beta = \beta\,, \tag{I2}$$

$\overset{\cdot}{\underset{\cdot}{\to}}$ is nonincreasing in the first argument and nondecreasing in the second, (I3)

$$\alpha \overset{\cdot}{\underset{\cdot}{\to}} \beta = \underset{S}{\neg}\beta \overset{\cdot}{\underset{\cdot}{\to}} \underset{S}{\neg}\alpha\,, \tag{I4}$$

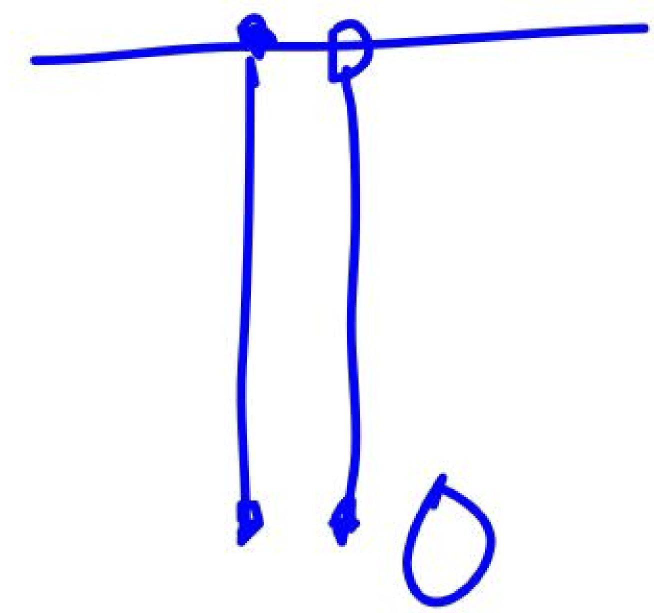
$$\alpha \overset{\cdot}{\underset{\cdot}{\to}} (\beta \overset{\cdot}{\underset{\cdot}{\to}} \gamma) = \beta \overset{\cdot}{\underset{\cdot}{\to}} (\alpha \overset{\cdot}{\underset{\cdot}{\to}} \gamma)\,, \tag{I5}$$

continuity. (I6)

R-implication (residuated fuzzy implication, residuum)

is an operation

$$\alpha \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup\{\gamma : \ \alpha \underset{\cdot}{\wedge} \gamma \leq \beta\}\,, \qquad \text{(RI)}$$



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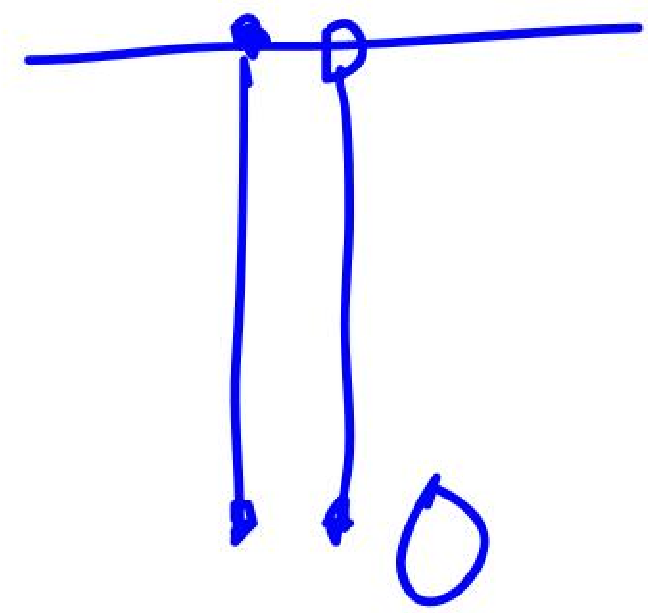

Examples of R-implications

- From the standard conjunction $\underset{S}{\wedge}$ we obtain the **Gödel implication**

$$\alpha \overset{R}{\underset{S}{\rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ \beta & \text{otherwise.} \end{cases}$$

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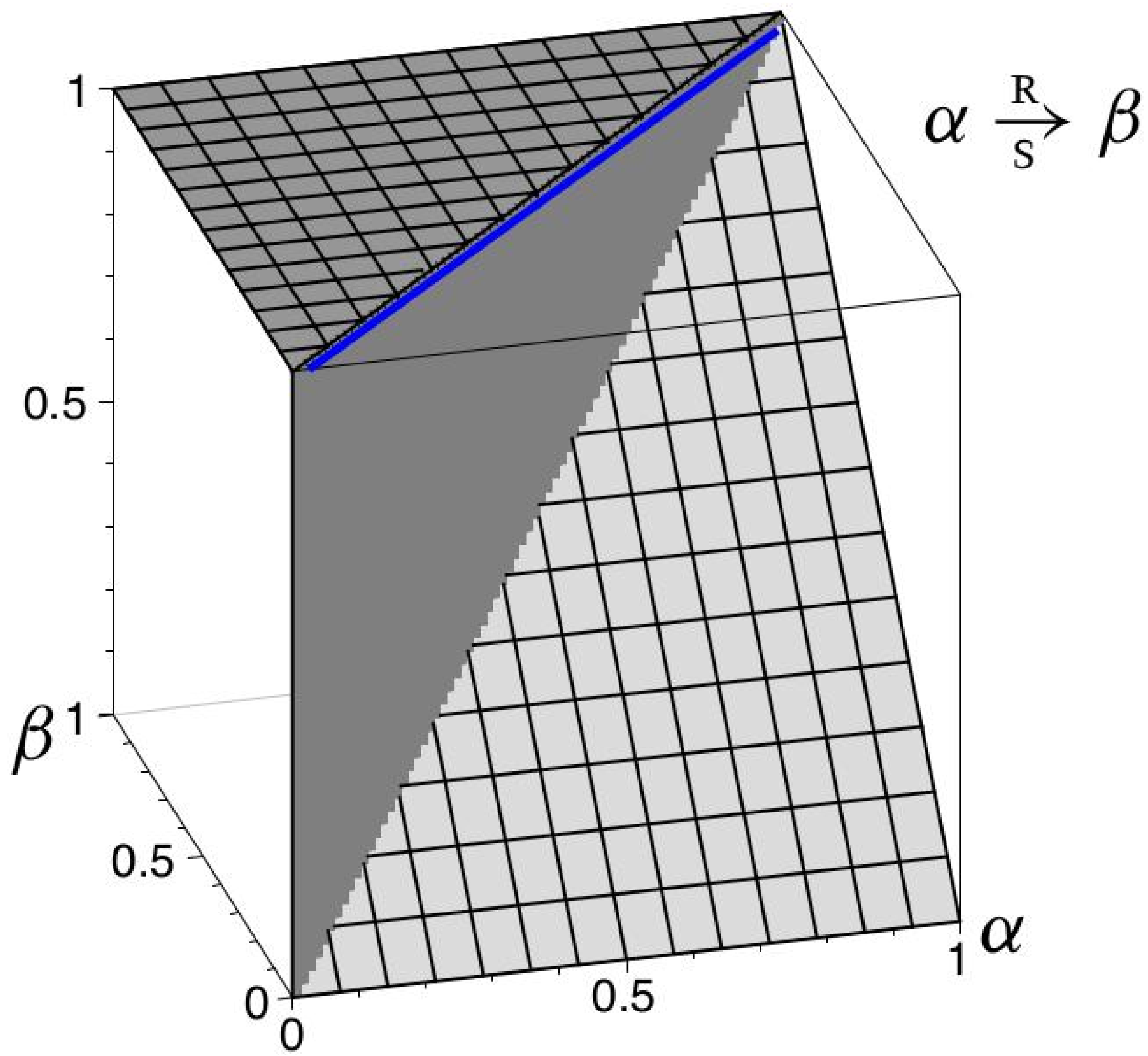

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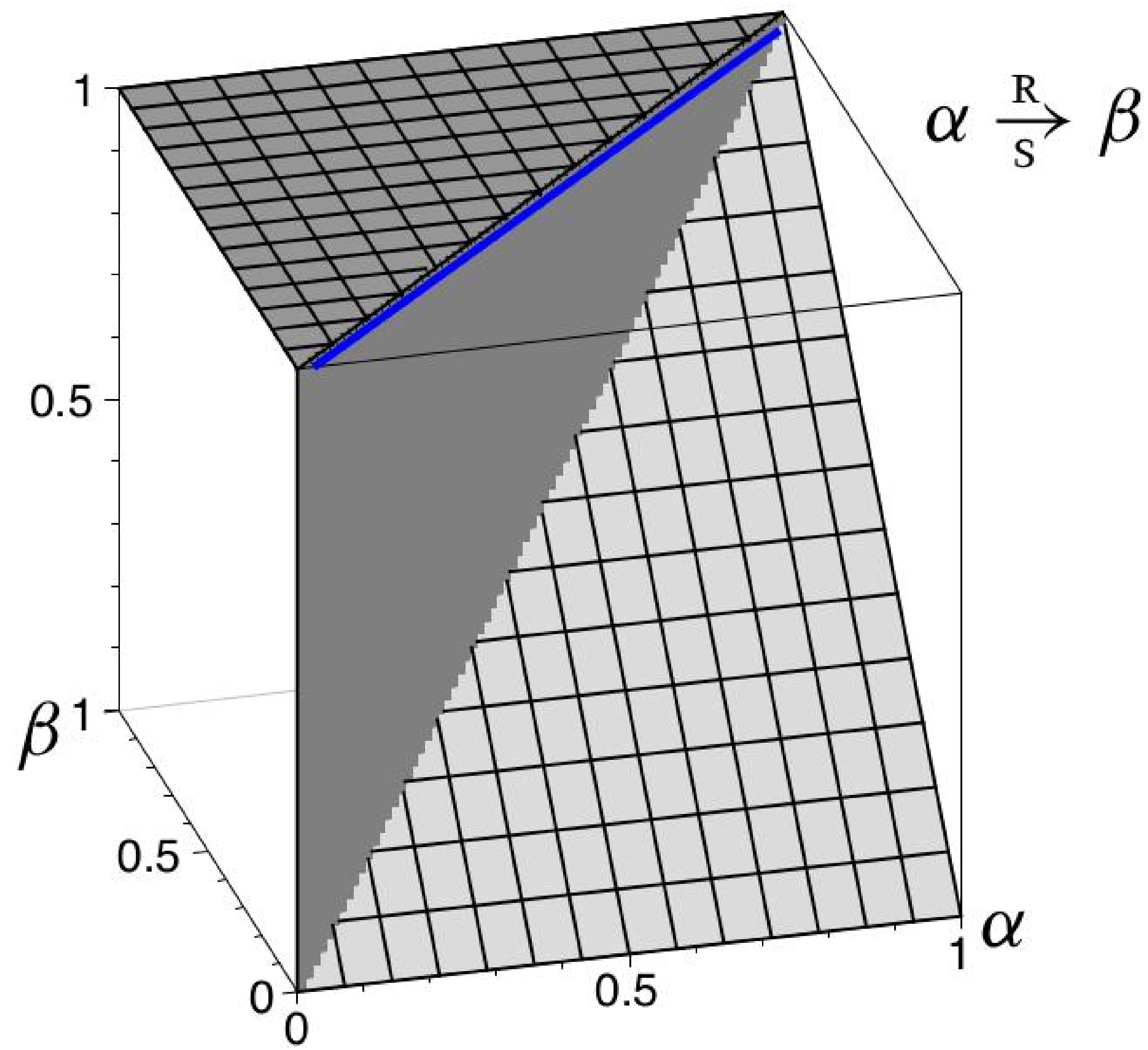

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$$\alpha \overset{\mathrm{R}}{\underset{\mathrm{L}}{\rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ 1 - \alpha + \beta & \text{otherwise.} \end{cases}$$

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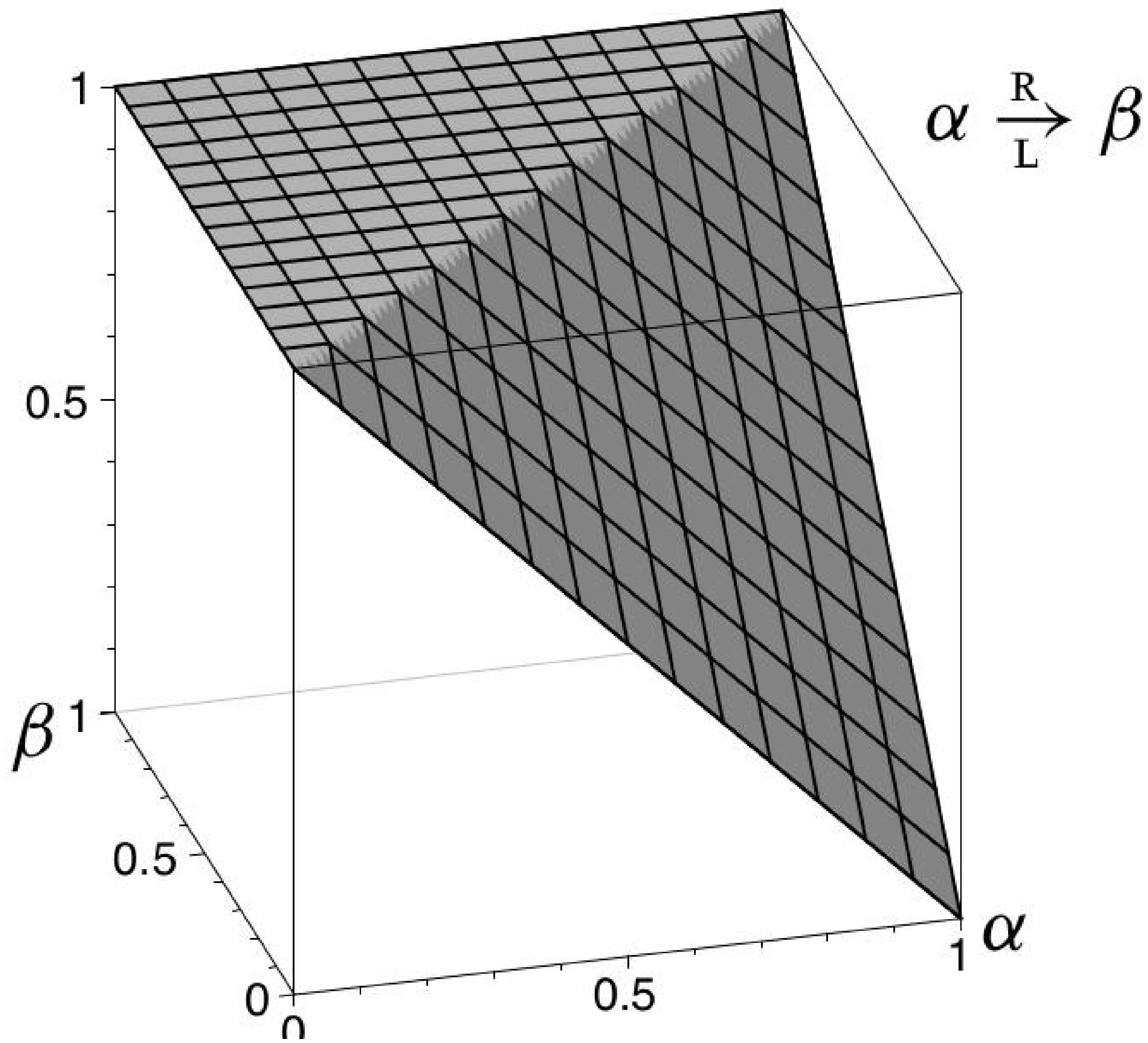

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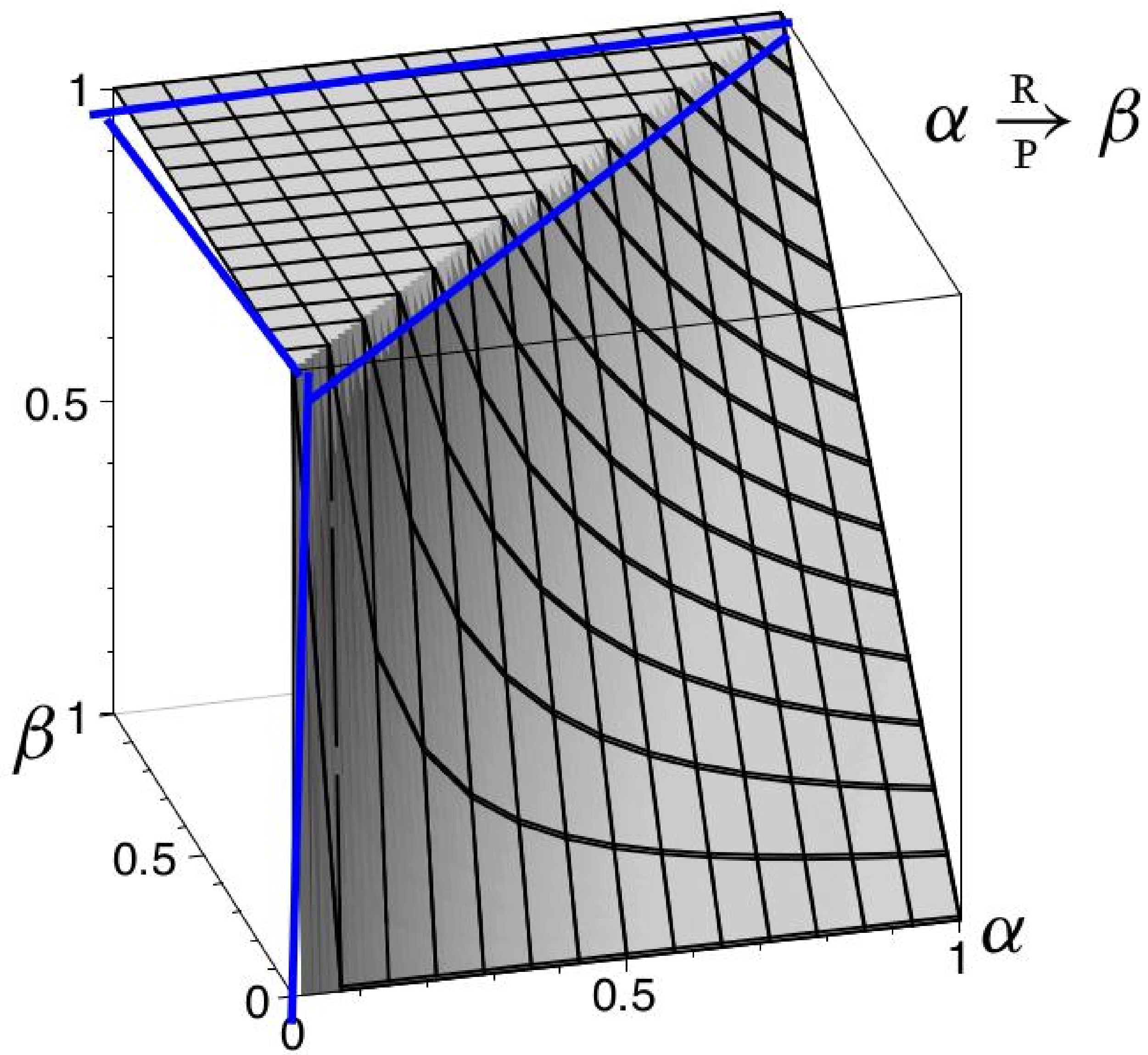

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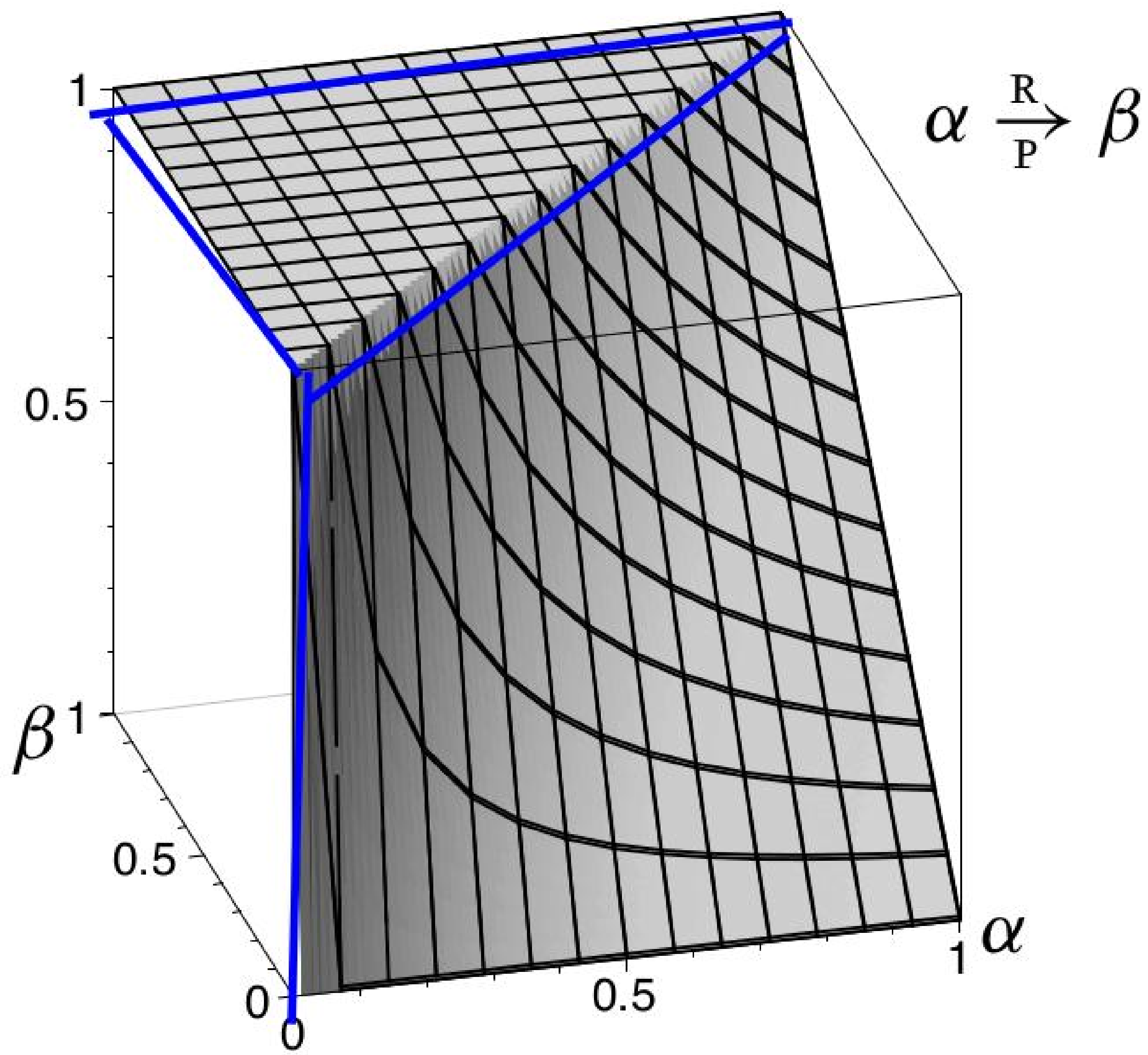

Properties of R-implications

Theorem: Let $\underset{\cdot}{\wedge}$ be a continuous fuzzy conjunction. Then the R-implication $\overset{\mathrm{R}}{\underset{\cdot}{\to}}$ satisfies (I1a), (I1b), (I2), (I3).

Proof: $\alpha \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup \Gamma(\alpha, \beta)$, where
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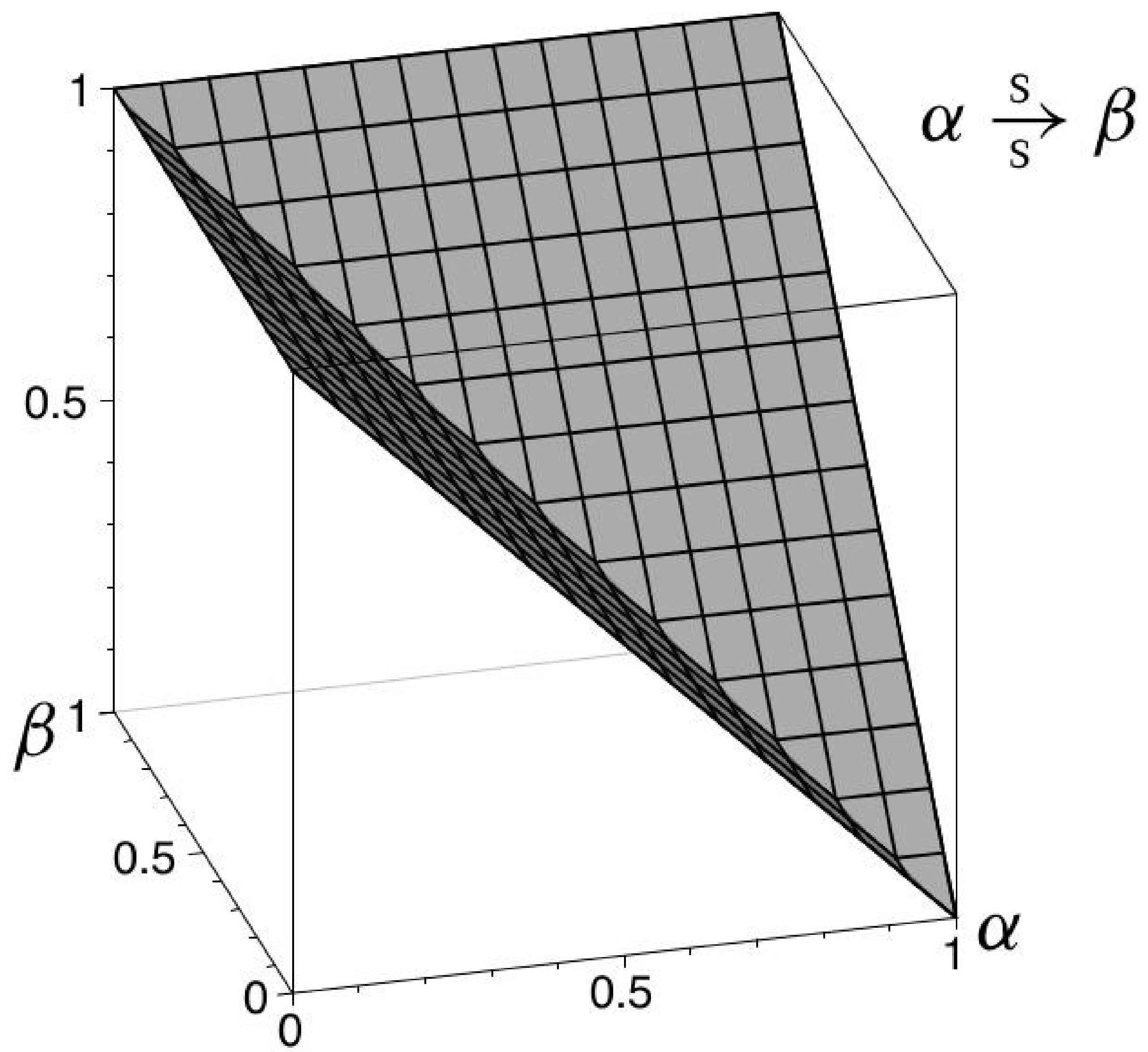

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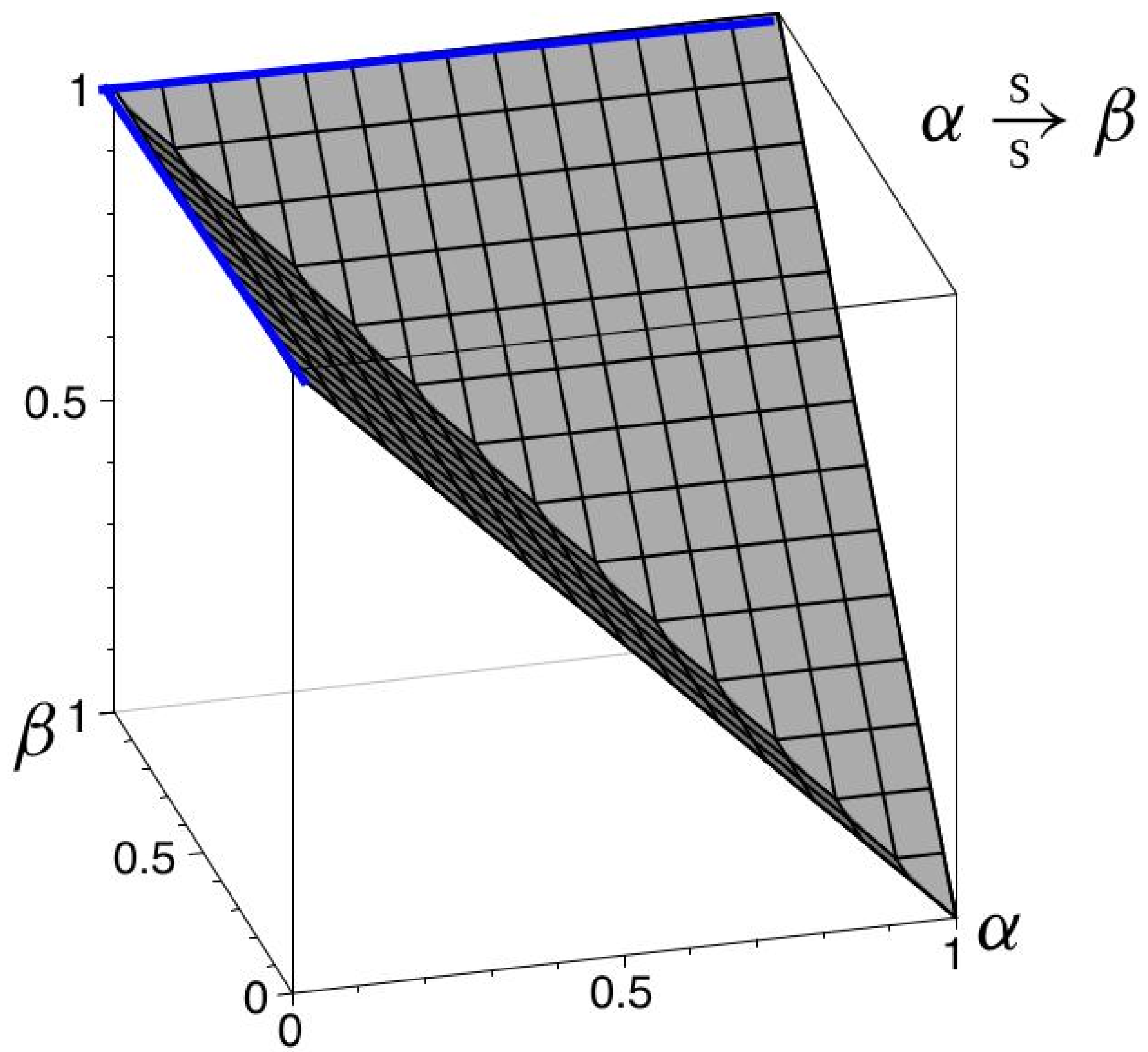

Examples of S-implications

- From the Łukasiewicz disjunction we obtain the **Łukasiewicz** implication $\underset{L}{\overset{S}{\rightarrow}}$ which

coincides with the Łukasiewicz residuated implication $\underset{L}{\overset{R}{\rightarrow}}$.

Among all fuzzy implications studied here, only residuated implications induced by nilpotent fuzzy conjunctions (e.g., the Łukasiewicz implication) satisfy all properties (I1a),(I1b),(I2)–(I6).

Fuzzy biimplication (equivalence)

is an operation $\underset{\cdot}{\dot{\leftrightarrow}}$ usually defined by

$$\alpha \underset{\cdot}{\dot{\leftrightarrow}} \beta = (\alpha \underset{\cdot}{\dot{\rightarrow}} \beta) \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\dot{\rightarrow}} \alpha),$$

where $\underset{\cdot}{\dot{\rightarrow}}$ is a fuzzy implication and $\underset{\cdot}{\wedge}$ is a fuzzy conjunction.
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If $\underset{\cdot}{\dot{\rightarrow}}$ satisfies (I1a) (e.g., for a residuated implication), at least one of the brackets equals 1, hence the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$ is irrelevant.

Example: **Łukasiewicz** biimplication: $\alpha \overset{\mathrm{R}}{\underset{\mathrm{L}}{\leftrightarrow}} \beta = 1 - |\alpha - \beta|.$

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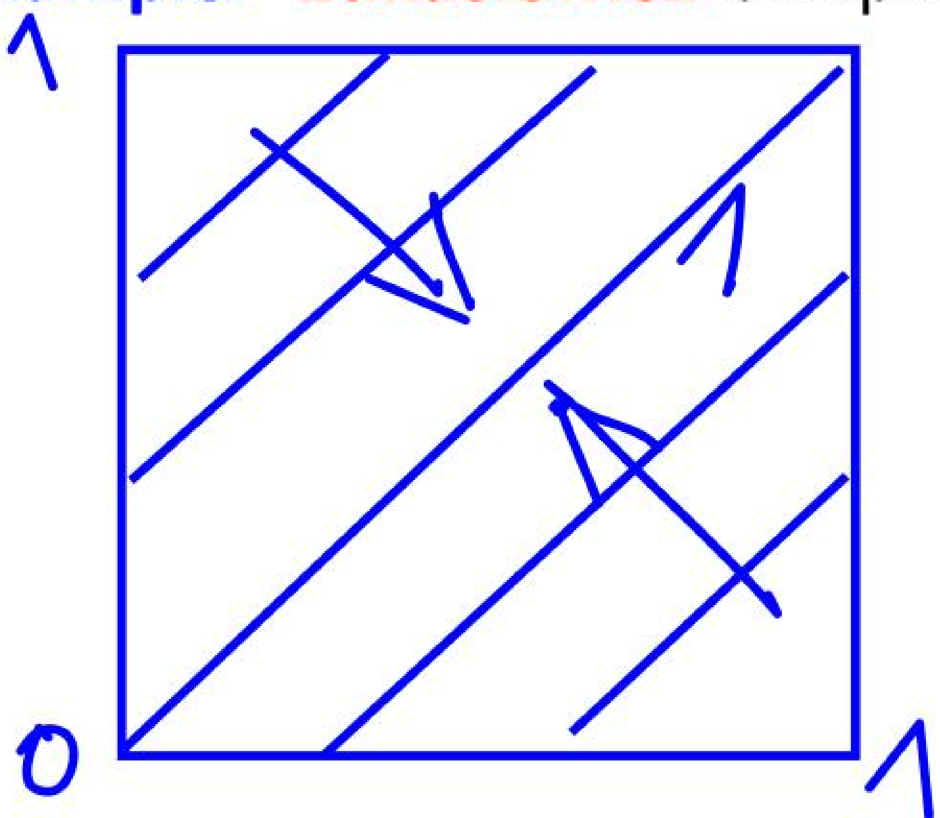

Classical relations

A **binary relation** is an $R \subseteq X \times Y$

Inverse relation to R: $R^{-1} \subseteq Y \times X$:

$$R^{-1} = \{(y, x) \in Y \times X : \ (x, y) \in R\}$$

The **composition** of relations $R \subseteq X \times Y$, $S \subseteq Y \times Z$ is $R \circ S \subseteq X \times Z$:

$$R \circ S = \Big\{(x, z) \in X \times Z : \ \big(\exists y \in Y : (x, y) \in R, (y, z) \in S\big)\Big\}$$

Using membership functions:

$$\mu_R : X \times Y \to \{0, 1\}$$
$$\mu_{R^{-1}}(y, x) = \mu_R(x, y)$$
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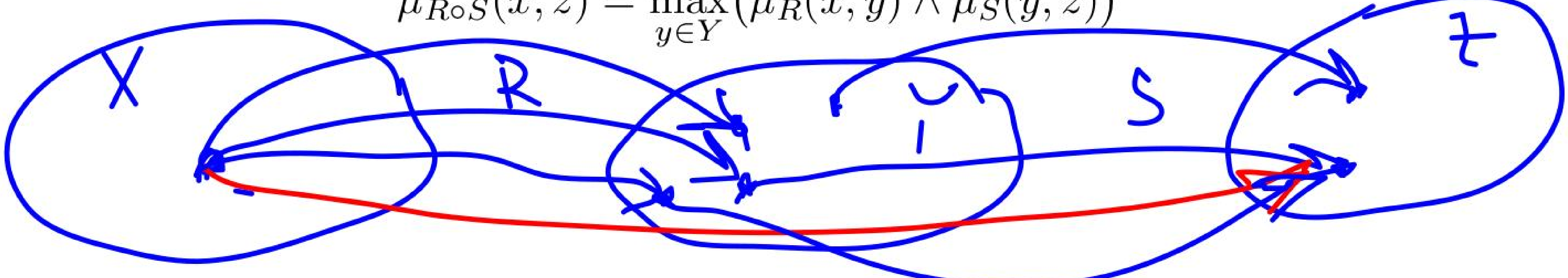

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<u>**Theorem**</u> The inversion of fuzzy relations is cut-consistent.

<u>**Theorem**</u> If Y is a finite set, then the standard composition of fuzzy relations $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ is cut-consistent.

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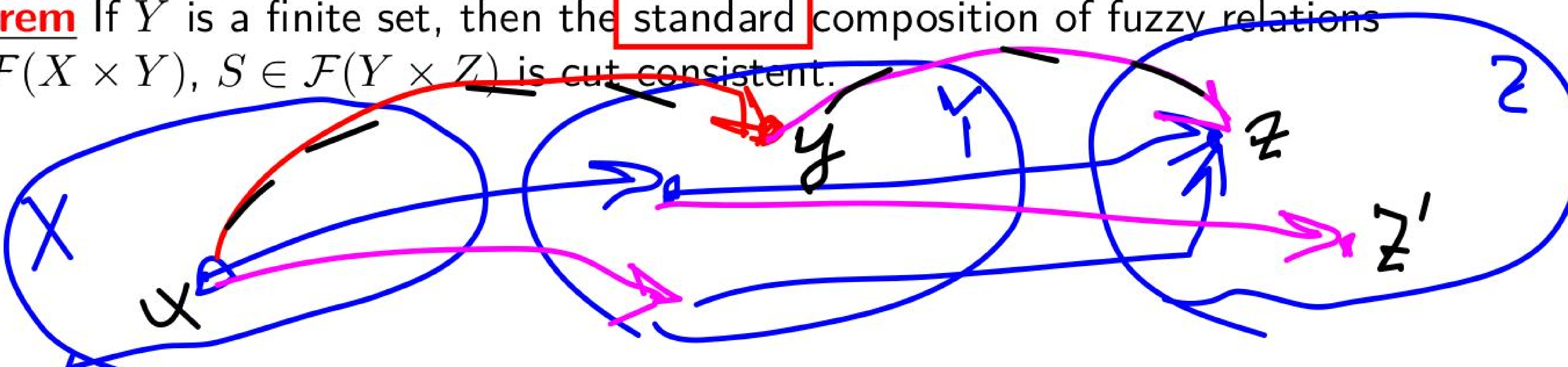

Special crisp relations

$R \subseteq X \times X$ can be:

- **an equality**: $E = \{(x, x) : x \in X\}$,
- **reflexive**: $\forall x \in X : (x, x) \in R$, i.e., $E \subseteq R$,
- **symmetric**: $(x, y) \in R \Rightarrow (y, x) \in R$, i.e., $R = R^{-1}$,
- **antisymmetric**: $\big((x, y) \in R\big) \wedge \big((y, x) \in R\big) \Rightarrow x = y$, i.e., $R \cap R^{-1} \subseteq E$,
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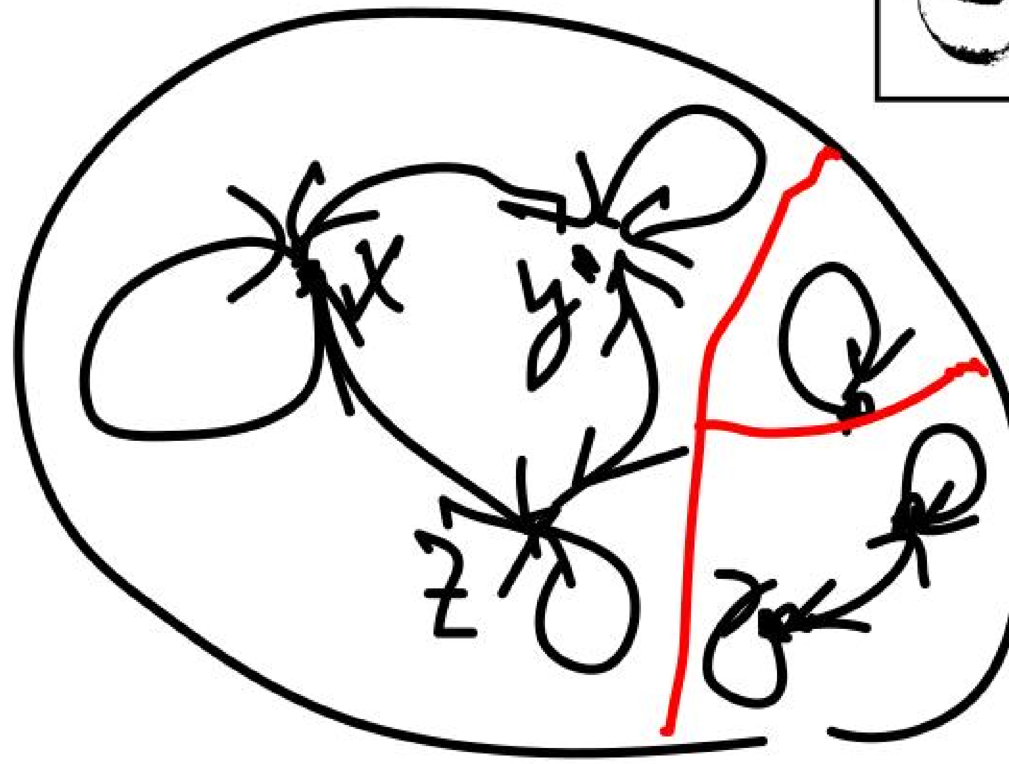

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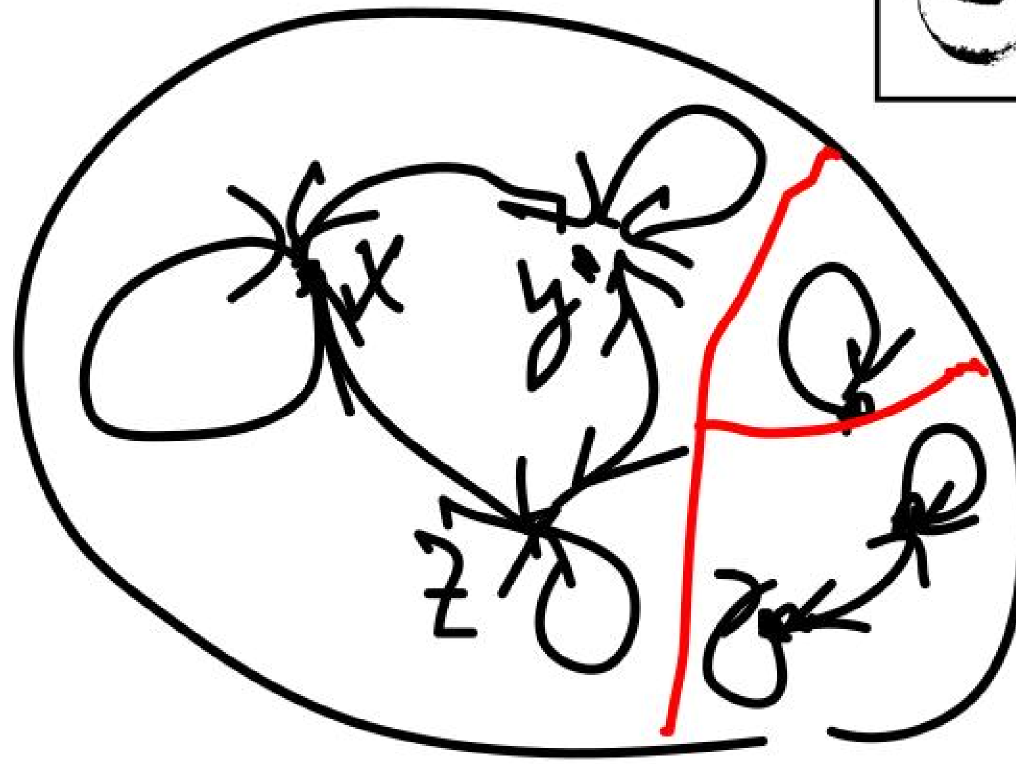

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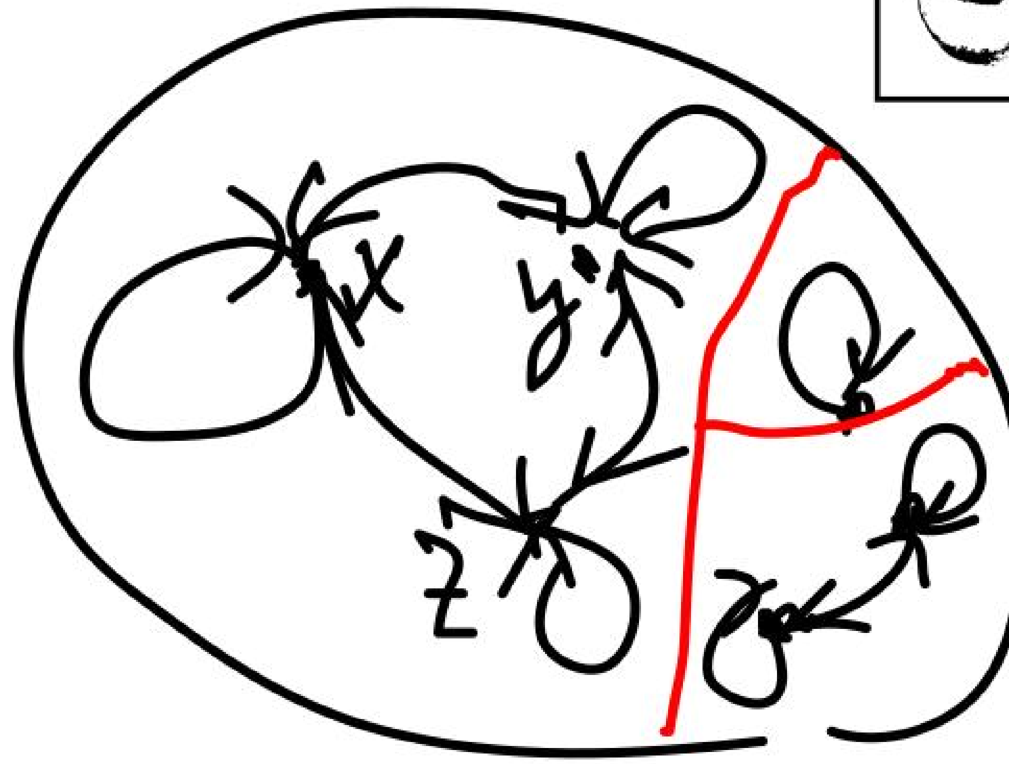

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Special fuzzy relations

A fuzzy relation $R \in \mathcal{F}(X \times X)$ can be:

- **reflexive**: $E \subseteq R$,
- **symmetric**: $R = R^{-1}$,
- **$\cdot$-antisymmetric**: $R \underset{\cdot}{\cap} R^{-1} \subseteq E$,
- **$\cdot$-transitive**: $R \underset{\cdot}{\circ} R \subseteq R$,
- **a $\cdot$-partial order**: $\cdot$-antisymmetric, reflexive, and $\cdot$-transitive,
- **an $\cdot$-equivalence**: symmetric, reflexive, and $\cdot$-transitive.

The last four terms depend on the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$.

Theorem The following properties of fuzzy relations are cut-consistent:

- reflexivity,
- symmetry,
- standard antisymmetry,
- product antisymmetry,
- standard transitivity,
- standard partial order,
- standard equivalence.

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$0.8 \wedge 0.7 \wedge 0.9 \leq \alpha$