Operations with crisp sets

set operations	propositional operations	formula
$\overline{} : \mathcal{P}(X) \to \mathcal{P}(X)$	$\neg : \{0,1\} \to \{0,1\}$	$\overline{A} = \big\{x \in X : \neg(x \in A)\big\}$
$\cap : \mathcal{P}(X)^2 \to \mathcal{P}(X)$	$\wedge : \{0,1\}^2 \to \{0,1\}$	$A \cap B = \big\{x \in X : (x \in A) \wedge (x \in B)\big\}$
$\cup : \mathcal{P}(X)^2 \to \mathcal{P}(X)$	$\vee : \{0,1\}^2 \to \{0,1\}$	$A \cup B = \big\{x \in X : (x \in A) \vee (x \in B)\big\}$

By means of membership functions:

$\mu_{\overline{A}}(x) = \neg \mu_A(x)$

$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x)$

$\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x)$

Laws of Boolean algebras

$$
\begin{aligned}
\neg\neg\alpha &= \alpha, & & \\
\alpha \vee \beta &= \beta \vee \alpha, & \alpha \wedge \beta &= \beta \wedge \alpha, \\
(\alpha \vee \beta) \vee \gamma &= \alpha \vee (\beta \vee \gamma), & (\alpha \wedge \beta) \wedge \gamma &= \alpha \wedge (\beta \wedge \gamma), \\
\alpha \wedge (\beta \vee \gamma) &= (\alpha \wedge \beta) \vee (\alpha \wedge \gamma), & \alpha \vee (\beta \wedge \gamma) &= (\alpha \vee \beta) \wedge (\alpha \vee \gamma), \\
\alpha \vee \alpha &= \alpha, & \alpha \wedge \alpha &= \alpha, \\
\alpha \vee (\alpha \wedge \beta) &= \alpha, & \alpha \wedge (\alpha \vee \beta) &= \alpha, \\
\alpha \vee 1 &= 1, & \alpha \wedge 0 &= 0, \\
\alpha \vee 0 &= \alpha, & \alpha \wedge 1 &= \alpha, \\
\alpha \wedge \neg\alpha &= 0, & \alpha \vee \neg\alpha &= 1, \\
\neg(\alpha \wedge \beta) &= \neg\alpha \vee \neg\beta, & \neg(\alpha \vee \beta) &= \neg\alpha \wedge \neg\beta.
\end{aligned}
$$

Fuzzy negation

unary operation $\underset{\cdot}{\neg} : [0,1] \to [0,1]$ such that

$$\alpha \leq \beta \Rightarrow \underset{\cdot}{\neg}\beta \leq \underset{\cdot}{\neg}\alpha, \tag{N1}$$

$$\underset{\cdot}{\neg}\underset{\cdot}{\neg}\alpha = \alpha. \tag{N2}$$

Example: **Standard negation**: $\underset{S}{\neg}\alpha = 1 - \alpha$.

Properties of fuzzy negations

Theorem: Each fuzzy negation $\underset{\cdot}{\neg}$ is a continuous, strictly decreasing bijection satisfying

$$\underset{\cdot}{\neg}\,1 = 0, \qquad \underset{\cdot}{\neg}\,0 = 1. \tag{N0}$$

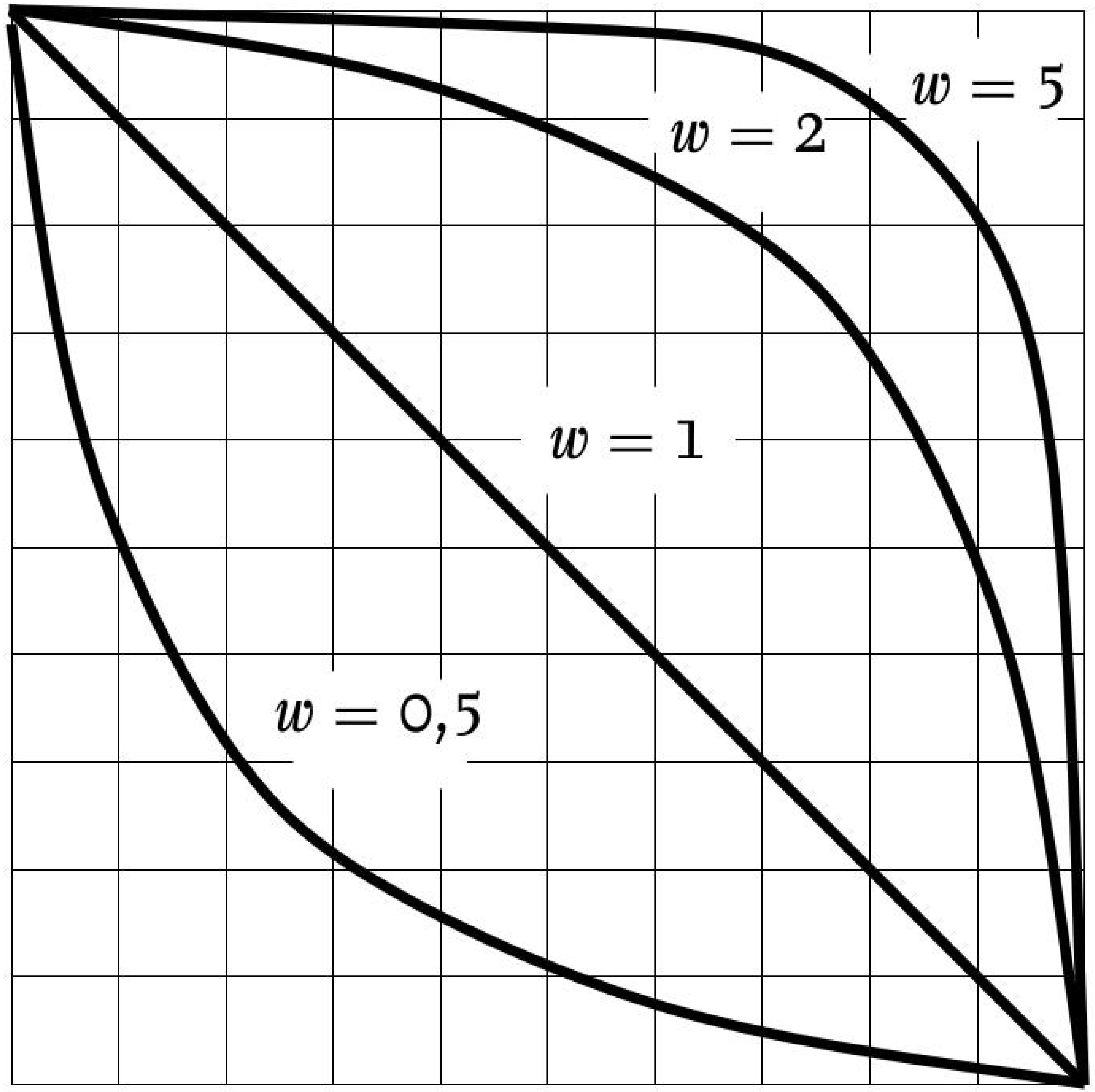
Its graph is symmetric w.r.t. the axis of the 1st and 3rd quadrant, i.e., $\underset{\cdot}{\neg}^{-1} = \underset{\cdot}{\neg}$

Proof:

- Injectivity: If $\underset{\cdot}{\neg}\,\alpha = \underset{\cdot}{\neg}\,\beta$, then $\alpha = \underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\alpha = \underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\beta = \beta$.
- Surjectivity: For each $\alpha \in [0,1]$ there is a $\beta \in [0,1]$ such that $\alpha = \underset{\cdot}{\neg}\,\beta$, namely $\beta = \underset{\cdot}{\neg}\,\alpha$.
- $\Rightarrow$ continuity and boundary conditions.
- The symmetry of the graph is equivalent to involutivity (N2).

Yager fuzzy negations

$$i(\alpha) = \alpha^w , \qquad i^{-1}(\alpha) = \alpha^{\frac{1}{w}} , \qquad \underset{\mathrm{Yw}}{\neg}\, \alpha = i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big) = (1-\alpha^w)^{\frac{1}{w}} , \qquad w \in (0,\infty)$$



Representation theorem for fuzzy negations

A function $\underset{\cdot}{\neg} : [0,1] \to [0,1]$ is a fuzzy negation iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**generator of fuzzy negation** $\underset{\cdot}{\neg}$) such that

$$\underset{\cdot}{\neg} = i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1}, \quad \text{i.e.,} \quad \underset{\cdot}{\neg}\,\alpha = i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big).$$

Proof: (According to [Nguyen-Walker].)

- Sufficiency:

(N1): Assume $\alpha, \beta \in [0,1]$, $\alpha \leq \beta$.
i, i^{-1} preserve the ordering, $\underset{\mathrm{S}}{\neg}$ reverses it:

$$\begin{aligned} i(\alpha) &\leq i(\beta) \\ \underset{\mathrm{S}}{\neg}\, i(\alpha) &\geq \underset{\mathrm{S}}{\neg}\, i(\beta) \\ i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big) &\geq i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\beta)\big) \\ \underset{\cdot}{\neg}\,\alpha &\geq \underset{\cdot}{\neg}\,\beta \end{aligned}$$

(N2): $\underset{\cdot}{\neg} \circ \underset{\cdot}{\neg} = i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} \circ i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} = i \circ \underset{\mathrm{S}}{\neg} \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} = i \circ i^{-1} = \mathrm{id}$,
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Possible construction of a generator of a fuzzy negation

- Necessity: We shall prove that

$$i(\alpha) = \frac{\alpha + \underset{S}{\neg}\,\underset{\cdot}{\neg}\,\alpha}{2}$$

is a generator of a fuzzy negation $\underset{\cdot}{\neg}$.
i is increasing, continuous, and satisfies $i(0) = 0$, $i(1) = 1$, thus i is a bijection on $[0, 1]$.

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A generator of a fuzzy negation is not unique.

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Fuzzy complement

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