Deduction theorem in classical logic

$\mathcal{T}$ ... theory

A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \to B$

Proof $\Leftarrow$:

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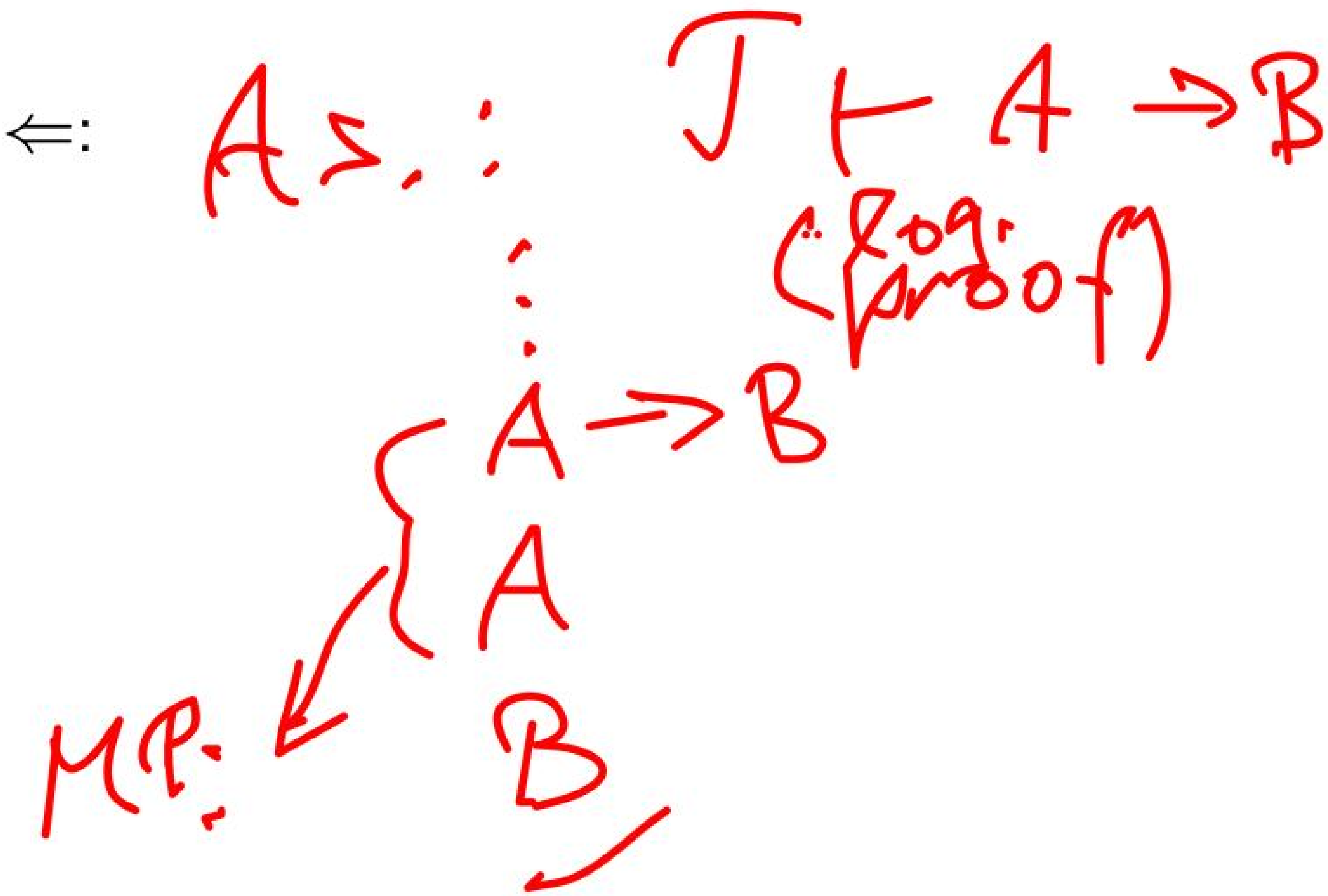

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Proof $\Leftarrow$:

$$
\begin{array}{rl}
 & //\text{BEGIN of proof of } \mathcal{T} \vdash A \to B \\
 & \vdots \\
 & D_{i-1} = A \to B \\
 & //\text{END of proof of } \mathcal{T} \vdash A \to B \\
\text{SA}: & D_i = A \\
\text{MP}(D_i, D_{i-1}): & D_{i+1} = B
\end{array}
$$

$\Rightarrow$: Proof by contradiction:

Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \rightarrow B$.

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Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \to B$.

$\vdots$

B

Case 1: B is an axiom (or pos. spec $\in \mathcal{T}$)

not!

Case 2: B = A $\qquad$ $\mathcal{T} \nvdash A \to A$

not!

Case 3: B resulted from deduction

C

$C \to B$ } apply DT

MP

B $\qquad$ shortest proof

$\Rightarrow$: Proof by contradiction:

Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \to B$.

1. B is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

$$\begin{aligned} & D_1 = B \\ RI(D_1): \quad & D_2 = A \to B \end{aligned}$$

hence $\mathcal{T} \vdash A \to B$.

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hence $\mathcal{T} \vdash A \to B$.

2. $B \neq A$ because $\mathcal{T} \vdash A \to A$.

3. B is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.

WLOG, we choose for B a formula with the shortest possible proof; its shortest proof must be of the following form:

$$\begin{array}{rl} & \vdots \\ & D_i \\ & \vdots \\ & D_j = D_i \to B \\ & \vdots \\ \text{MP}(D_i, D_j): & D_m = B \end{array}$$

for $i < j < m$ or $j < i < m$.

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for $i < j < m$ or $j < i < m$.

The proofs of $\mathcal{T} \cup \{A\} \vdash D_i$, $\mathcal{T} \cup \{A\} \vdash D_j$ are of lengths $< m$, therefore

$$\mathcal{T} \vdash A \to D_i$$

$$\mathcal{T} \vdash A \to D_j = A \to (D_i \to B)$$

Proof of $\mathcal{T} \vdash A \to B$:

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\begin{array}{rl}
 & \text{//BEGIN of proof of } \mathcal{T} \vdash A \to D_i \\
 & \vdots \\
 & D_k = A \to D_i \\
 & \text{//END of proof of } \mathcal{T} \vdash A \to D_i \\
 & \text{//BEGIN of proof of } \mathcal{T} \vdash A \to D_j \\
 & \vdots \\
 & D_n = A \to \overbrace{(D_i \to B)}^{D_j} \\
 & \text{//END of proof of } \mathcal{T} \vdash A \to D_j \\
\text{(C2) } B := D_i, C := B: & D_{n+1} = (A \to (D_i \to B)) \to ((A \to D_i) \to (A \to B)) \\
\text{MP}(D_n, D_{n+1}): & D_{n+2} = (A \to D_i) \to (A \to B) \\
\text{MP}(D_k, D_{n+2}): & D_{n+3} = A \to B
\end{array}
$$

Corollary Cor2 $A \vdash A \vee B$ for all A, B

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$$A \vdash \neg A \rightarrow B = A \vee B$$

$$\Updownarrow \text{(DT)}$$

$$\text{ALL}(A): \quad \{A, \neg A\} \vdash B$$

$\Rightarrow$ we can add a deduction rule $\dfrac{A}{A \vee B}$ (and $\dfrac{B}{A \vee B}$ was already proved in Cl9)

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Corollary Cor3 $A \vdash \neg\neg A$, $\vdash A \to \neg\neg A$ for all A

$(A \to 0) \to 0$

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$$\vdash A \to \overbrace{(\neg A \to \mathbf{0})}^{\neg\neg A}$$

$$\Updownarrow \text{(DT)}$$

$$A \vdash \neg A \to \mathbf{0}$$

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Corollary Cor4 $\neg\neg A \vdash A$, $\quad \vdash \neg\neg A \to A$ for all A

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$$\vdash A \to ((A \to 0) \to 0)$$

$$A \vdash (A \to 0) \to 0 = \neg A \to 0$$

$$\{A, \neg A\} \vdash 0$$

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Corollary Cor5 $\vdash A \leftrightarrow \neg\neg A$ (can be added to axioms)

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How can we simplify our proofs?

$B \leftrightarrow C \vdash (A \rightarrow B) \leftrightarrow (A \rightarrow C)$

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INTERPLAY OF SYNTAX AND SEMANTICS OF CLASSICAL LOGIC

Weak soundness Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.

Strong soundness For any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness Each tautology is provable, i.e., if $\models A$, then $\vdash A$.

Strong completeness For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$.

BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

$\mathcal{A}$ ... countable set of propositional variables

$\mathcal{L} = \{\rightarrow, \mathbf{0}, \wedge\}$ ... the set of logical connectives:

$\rightarrow$ ... (binary) implication

$\mathbf{0}$ ... (nulary) false

$\wedge$ ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:

$\neg A = A \rightarrow \mathbf{0}$ ... (unary) negation

$\mathbf{1} = \neg\mathbf{0} = \mathbf{0} \rightarrow \mathbf{0}$ ... (nulary) true

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$A \underset{\mathsf{S}}{\wedge} B = A \wedge (A \rightarrow B)$

$A \overset{\mathsf{S}}{\vee} B = ((A \rightarrow B) \rightarrow B) \underset{\mathsf{S}}{\wedge} ((B \rightarrow A) \rightarrow A)$

no $A \vee B$ in general

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SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

Standard semantics the set of truth values ... $[0,1]$

$\wedge$ ... continuous fuzzy conjunction $\underset{\cdot}{\wedge}$

$\rightarrow$ ... residuum $\underset{\cdot}{\rightarrow}$ of $\underset{\cdot}{\wedge}$

$\mathbf{0}$ ... 0

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:

$\neg$ ... $\underset{\cdot}{\neg}$, where $\underset{\cdot}{\neg}\,\alpha = \alpha \underset{\cdot}{\rightarrow} 0$

$\mathbf{1}$ ... 1

$\leftrightarrow$ ... $\underset{\cdot}{\leftrightarrow}$, where $\alpha \underset{\cdot}{\leftrightarrow} \beta = (\alpha \underset{\cdot}{\rightarrow} \beta) \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\rightarrow} \alpha)$

$\underset{S}{\wedge}$ ... $\underset{S}{\wedge} = \min$

$\overset{S}{\vee}$ ... $\overset{S}{\vee} = \max$

Exercise Verify that the interpretation of $\underset{S}{\wedge}, \overset{S}{\vee}$ is independent of the choice of the fuzzy conjunction.

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An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\wedge$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:
1-tautology is a formula A which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\wedge$ and its residuum as an interpretation of $\rightarrow$)

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$,
$\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T} : e(B) = 1$.