Representation theorem for strict fuzzy conjunctions

Operation $\underset{\cdot}{\wedge}\colon [0,1]^2 \to [0,1]$ is a strict fuzzy conjunction iff there is an increasing bijection $i\colon [0,1] \to [0,1]$ (**multiplicative generator**) such that

$$\alpha \underset{\cdot}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{\mathrm{P}}{\wedge} i(\beta)\big) = i^{-1}\big(i(\alpha) \cdot i(\beta)\big)\,.$$

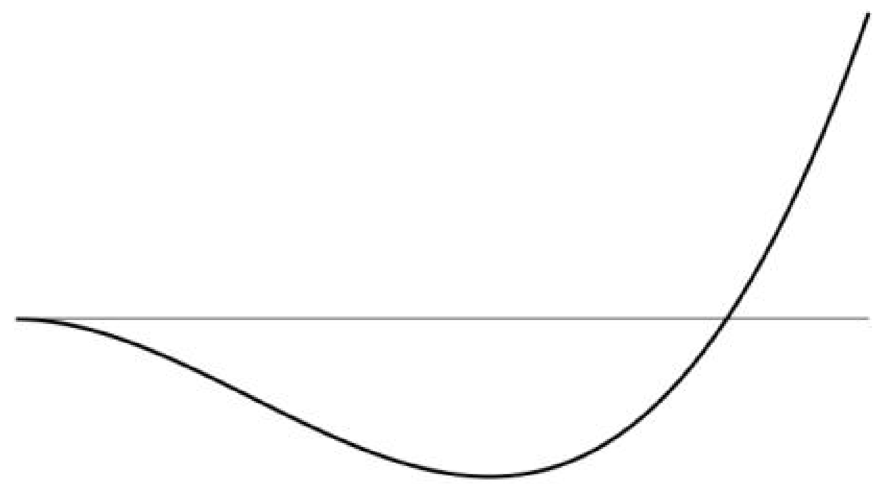
Sufficiency has been already proved.
The proof of necessity is much more advanced.

A multiplicative generator of a strict fuzzy conjunction is not unique.

Inspiration

Given a function, we can estimate the "shape" of its derivative or integral

function



derivative

integral

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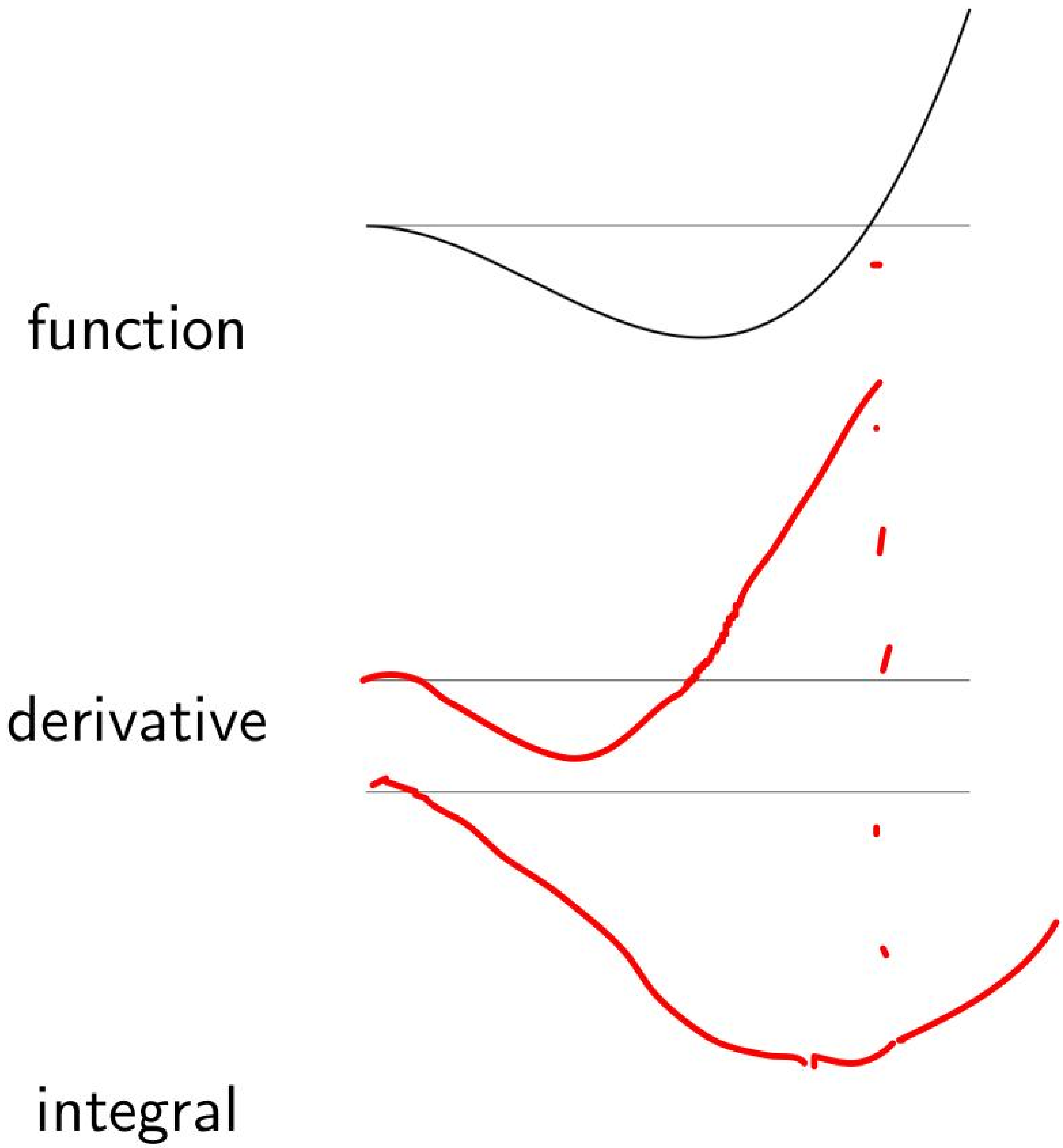

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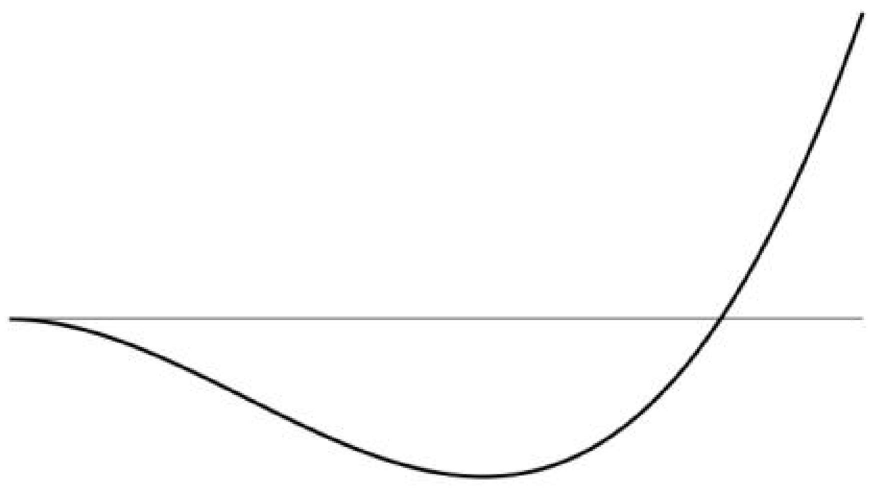

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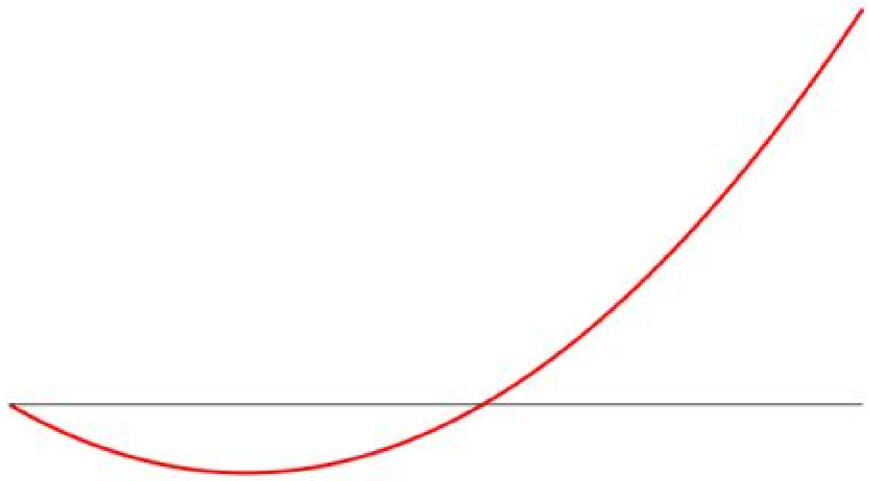
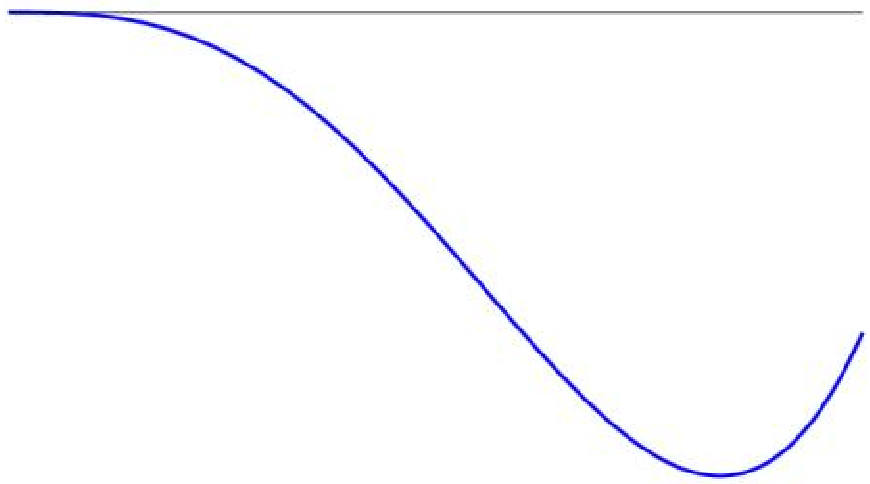
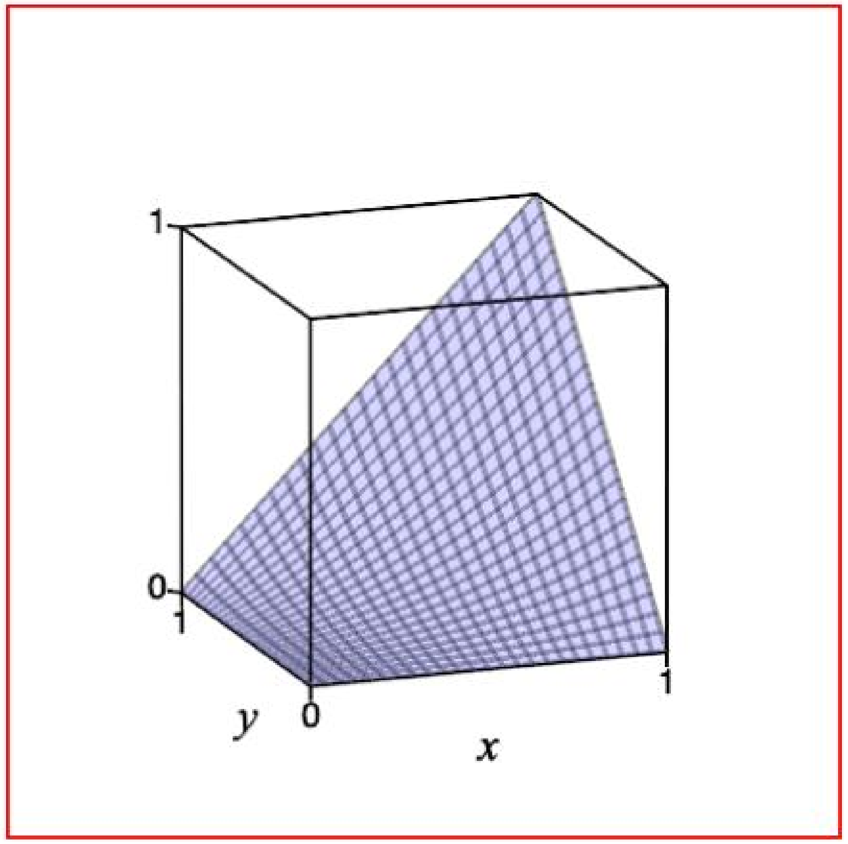
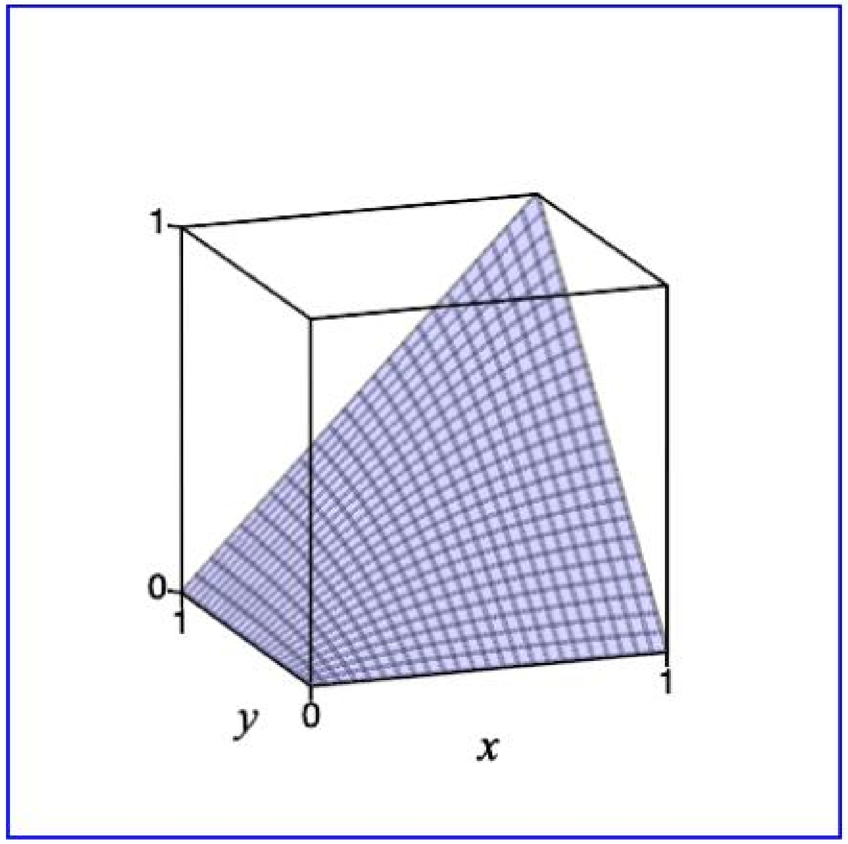


integral

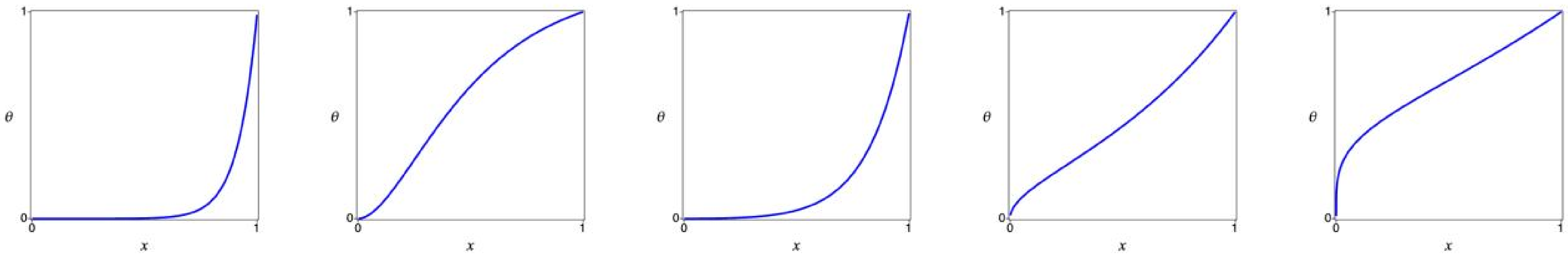


Comparing the “shape” of t-norms and their generators





Which of the multiplicative generators below corresponds to which of the t-norms above?



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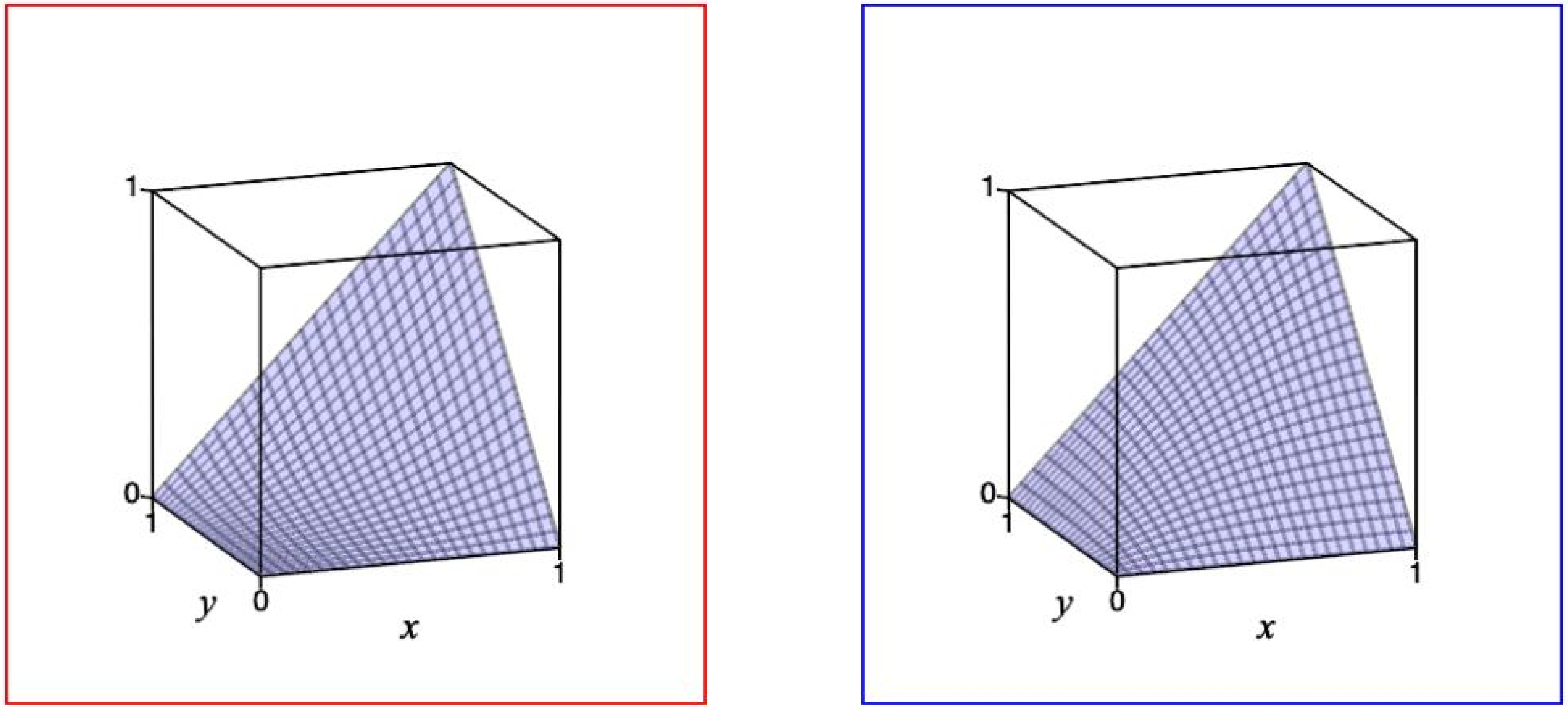

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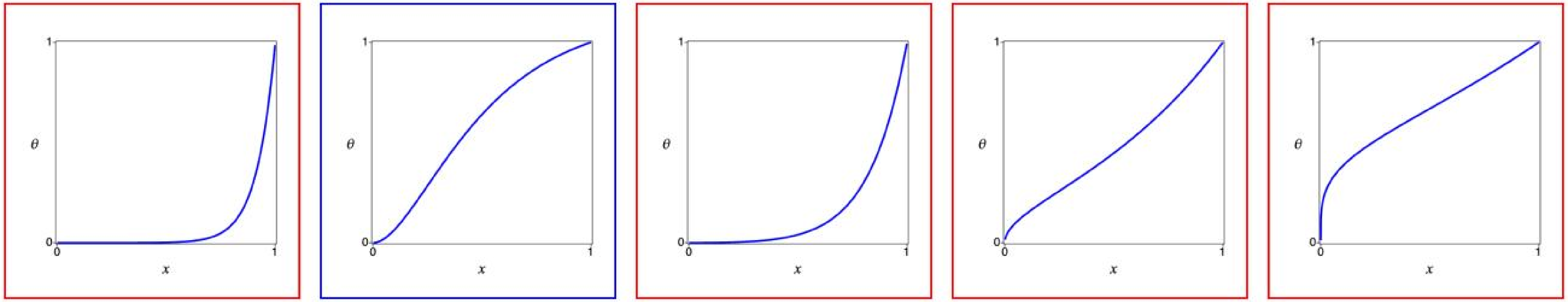

Representation theorem for strict fuzzy conjunctions

Simple proof of a special case:
We denote by

$$\alpha \mathbin{\dot\wedge}' \beta = \lim_{\varepsilon \to 0+} \frac{\big(\alpha \mathbin{\dot\wedge} (\beta + \varepsilon)\big) - (\alpha \mathbin{\dot\wedge} \beta)}{\varepsilon} = \frac{\partial}{\partial \beta}(\alpha \mathbin{\dot\wedge} \beta)$$

the right partial derivative of $\dot\wedge$ w.r.t. the second variable, evaluated at (α, β).

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provided that $0 < i'(0) < \infty$.

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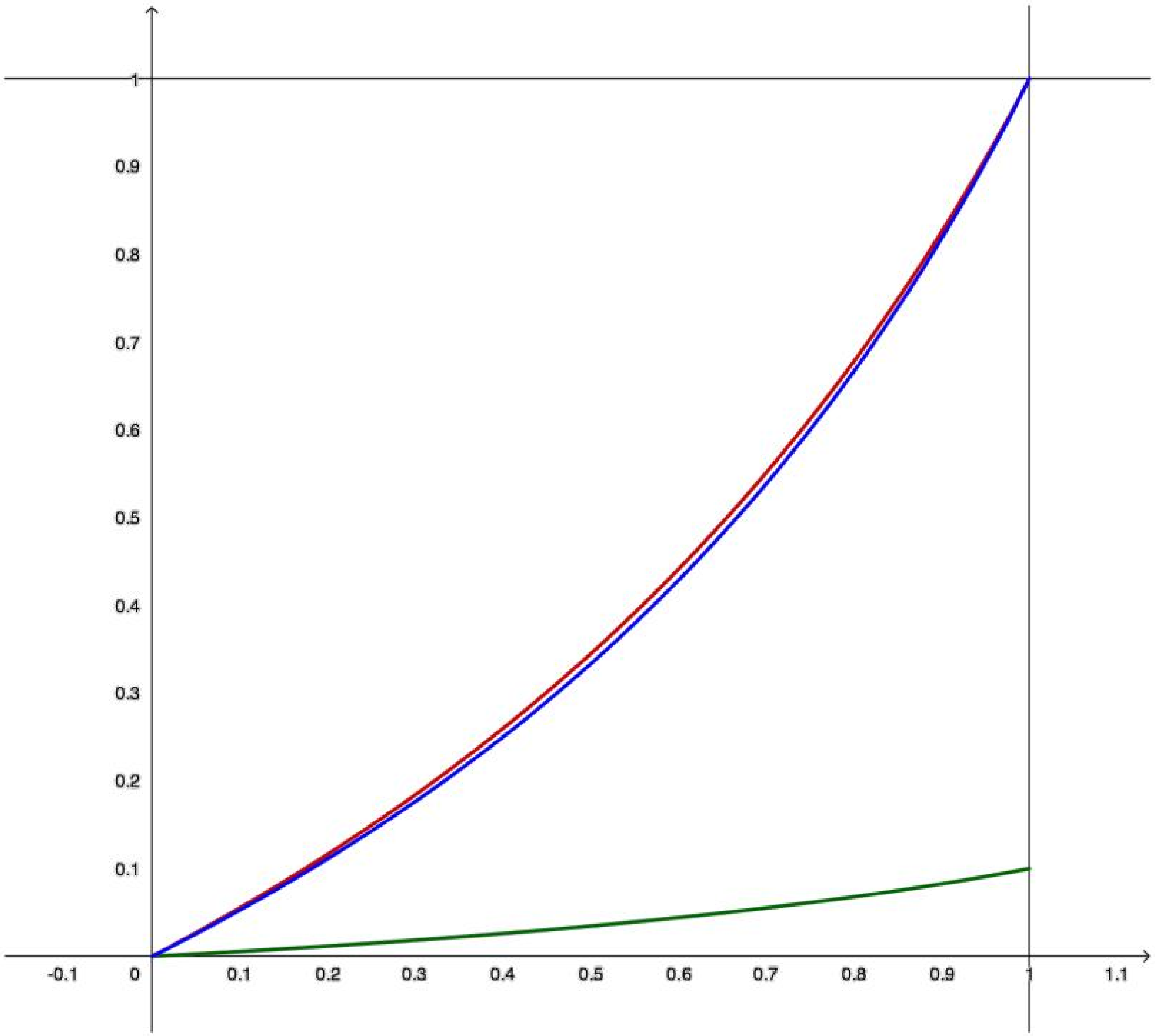

Horizontal: α, green: $\alpha \mathbin{\dot\wedge} \varepsilon$, red: $\dfrac{\alpha \mathbin{\dot\wedge} \varepsilon}{\varepsilon}$, blue: generator $i(\alpha) = \lim_{\varepsilon \to 0+} \dfrac{\alpha \mathbin{\dot\wedge} \varepsilon}{\varepsilon}$.

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Theorem: The class $[\wedge_{\mathrm{P}}]$ is the set of all strict conjunctions.

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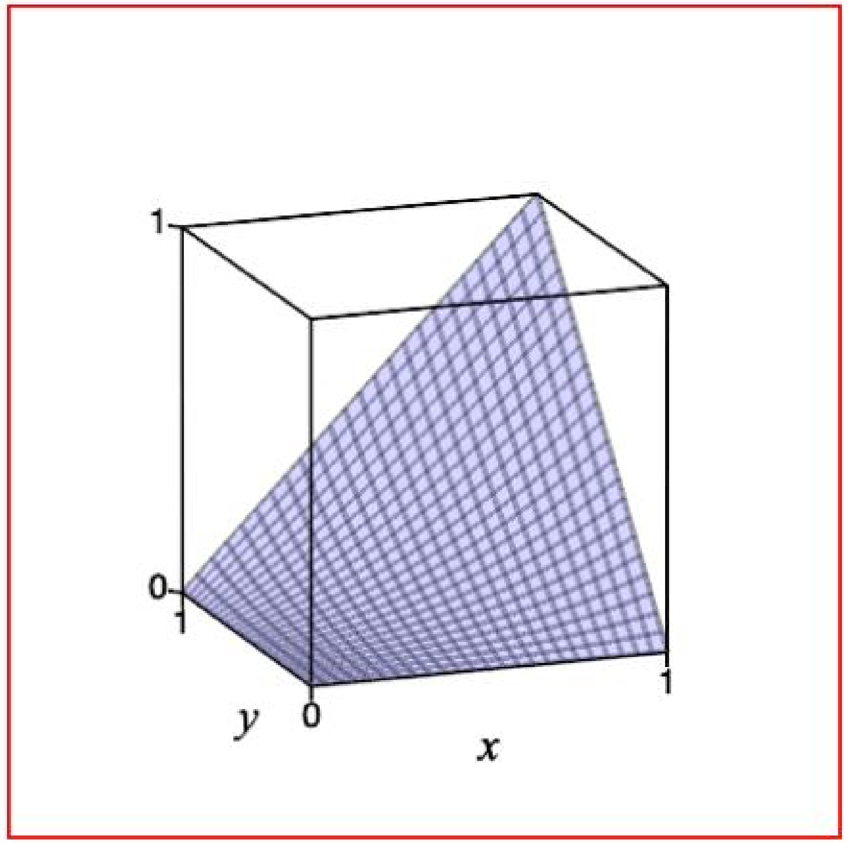

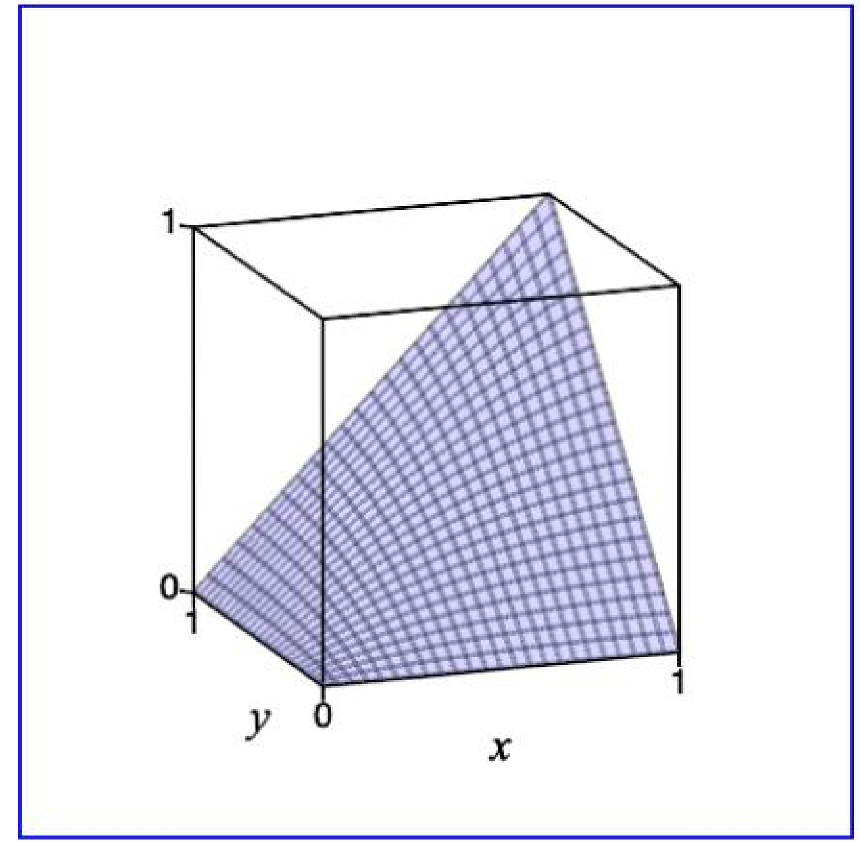

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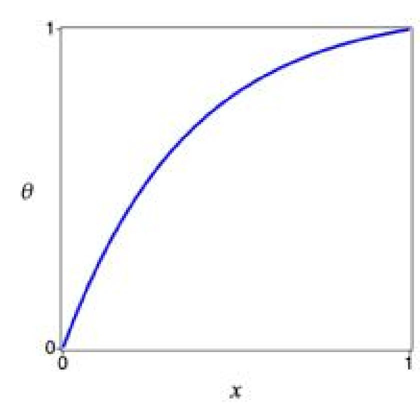

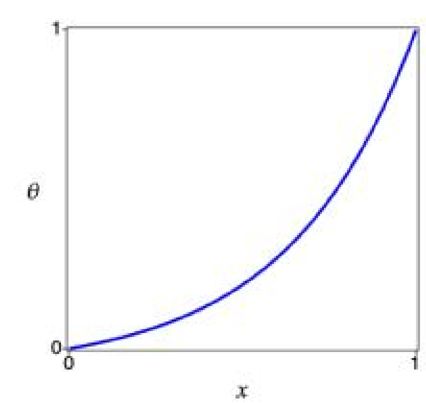

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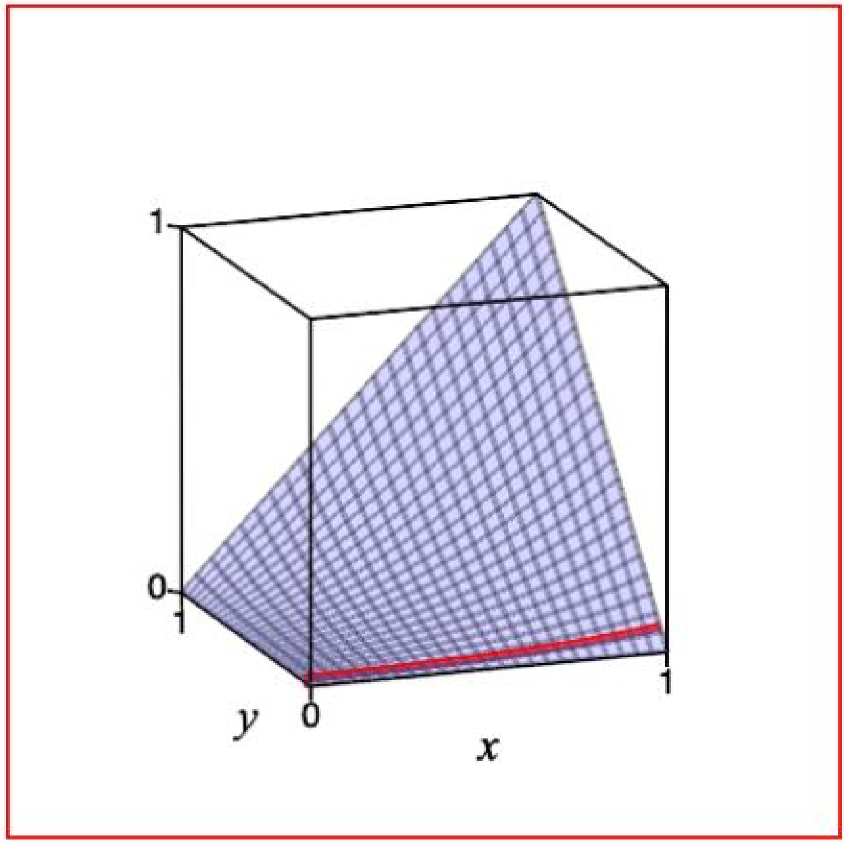

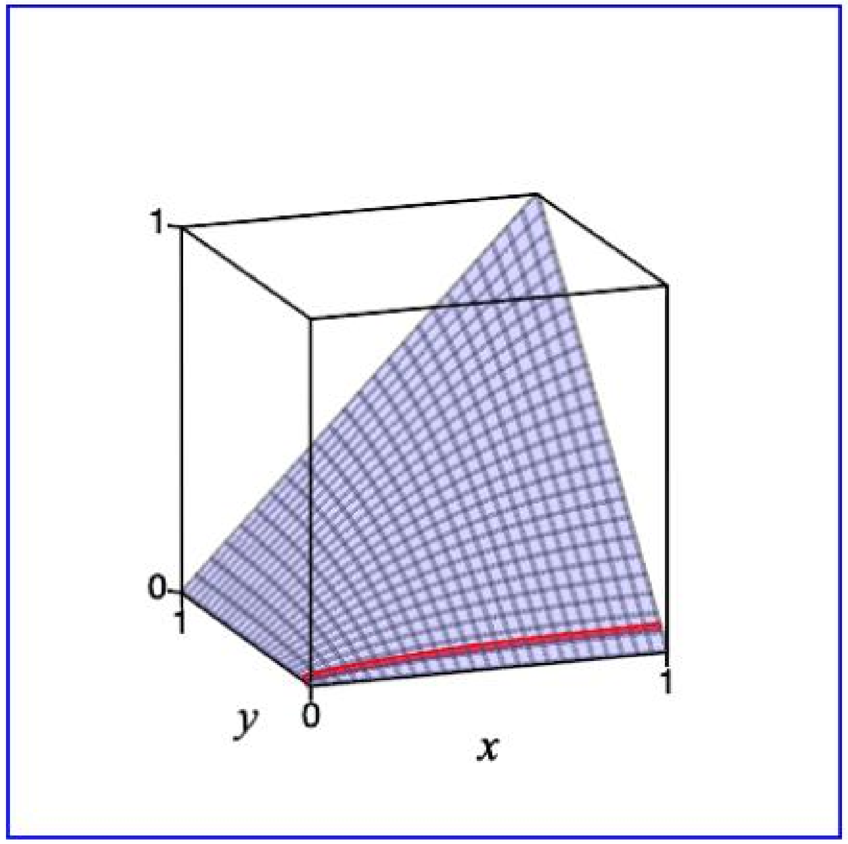

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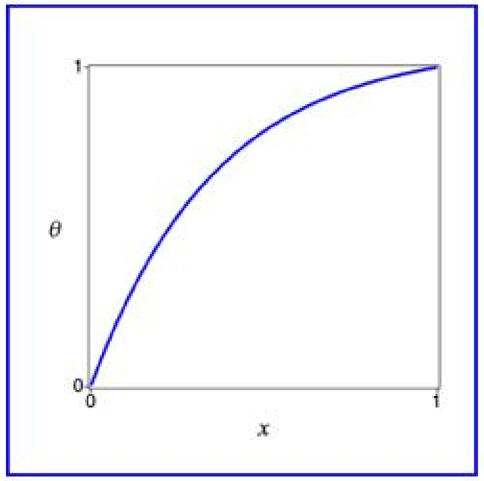

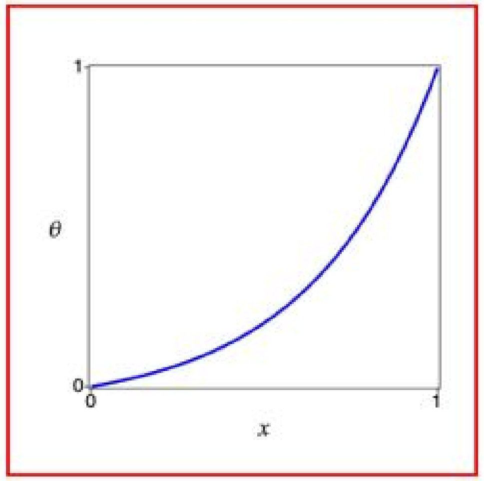

Representation theorem for nilpotent fuzzy conjunctions

Operation $\mathbin{\underset{\cdot}{\wedge}}\colon [0,1]^2 \to [0,1]$ is a **nilpotent** fuzzy conjunction iff there is an increasing bijection $i\colon [0,1] \to [0,1]$ (**Łukasiewicz generator**) such that

$$\alpha \mathbin{\underset{\cdot}{\wedge}} \beta = i^{-1}\big(i(\alpha) \mathbin{\underset{\mathrm{L}}{\wedge}} i(\beta)\big).$$

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Summary:

Theorem: The class $[\underset{\text{L}}{\wedge}]$ is the set of all nilpotent conjunctions.

Application of the representation theorem for nilpotent fuzzy conjunctions

Theorem: Let $\dot\wedge$ be a **nilpotent** fuzzy conjunction. Then

$$\forall \alpha \in (0,1)\ \exists n \in \mathbb{N} : \dot{\bigwedge}_{k=1}^{n} \alpha = 0$$

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Proof: According to the representation theorem, it suffices (without loss of generality) to prove the theorem for the Łukasiewicz conjunction. For a sufficiently large n we obtain

$$\alpha + \sum_{i=2}^{n} (\alpha - 1) \leq 0, \qquad \bigwedge\nolimits_{\mathrm{L}\; k=1}^{\;n} \alpha = 0.$$

Fuzzy intersection

is an operation on fuzzy sets defined using a fuzzy conjunction:

$$\mu_{A \underset{\cdot}{\cap} B}(x) = \mu_A(x) \underset{\cdot}{\wedge} \mu_B(x)$$

(we distinguish them by the same indices as the respective fuzzy conjunctions)

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Proof: 1. Cutworthiness:

$$\begin{aligned}
\mathcal{R}_{A \underset{S}{\cap} B}(\alpha) &= \{x \in X : \mu_{A \underset{S}{\cap} B}(x) \geq \alpha\} \\
&= \{x \in X : (\mu_A(x) \geq \alpha) \wedge (\mu_B(x) \geq \alpha)\} \\
&= \{x \in X : \mu_A(x) \geq \alpha\} \cap \{x \in X : \mu_B(x) \geq \alpha\} \\
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2. Cuts $\mathcal{R}_A(\alpha) \cap \mathcal{R}_B(\alpha)$ (for all $\alpha \in (0, 1]$) determine a unique fuzzy set equal to $A \underset{S}{\cap} B$.

Fuzzy disjunction (triangular conorm, t-conorm)

is a binary operation $\dot\vee\colon [0,1]^2 \to [0,1]$ such that

$\alpha \mathbin{\dot\vee} \beta = \beta \mathbin{\dot\vee} \alpha$	(commutativity)	(S1)
$\alpha \mathbin{\dot\vee} (\beta \mathbin{\dot\vee} \gamma) = (\alpha \mathbin{\dot\vee} \beta) \mathbin{\dot\vee} \gamma$	(associativity)	(S2)
$\beta \le \gamma \Rightarrow \alpha \mathbin{\dot\vee} \beta \le \alpha \mathbin{\dot\vee} \gamma$	(monotonicity)	(S3)
$\alpha \mathbin{\dot\vee} 0 = \alpha$	(boundary condition)	(S4)

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$$\alpha \mathbin{\dot{\vee}} 0 = \alpha \qquad \text{(boundary condition)} \qquad \text{(S4)}$$

Proposition: $\alpha \mathbin{\dot{\vee}} 1 = 1$.

Proof: $\alpha \mathbin{\dot{\vee}} 1 \overset{\text{(S3)}}{\geq} 0 \mathbin{\dot{\vee}} 1 \overset{\text{(S4)}}{=} 1$.

Examples of fuzzy disjunctions

- **Standard** (**max**, **Gödel**, **Zadeh** . . .):

$$\alpha \overset{\mathrm{S}}{\vee} \beta = \max(\alpha, \beta).$$

- **Product** (**probabilistic** . . .):

$$\alpha \overset{\mathrm{P}}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta.$$

- **Łukasiewicz** (**Giles**, **bold**, **bounded sum** . . .):

$$\alpha \overset{\mathrm{L}}{\vee} \beta = \begin{cases} \alpha + \beta & \text{for } \alpha + \beta < 1, \\ 1 & \text{otherwise.} \end{cases}$$

- **Drastic** (**weak** . . .):

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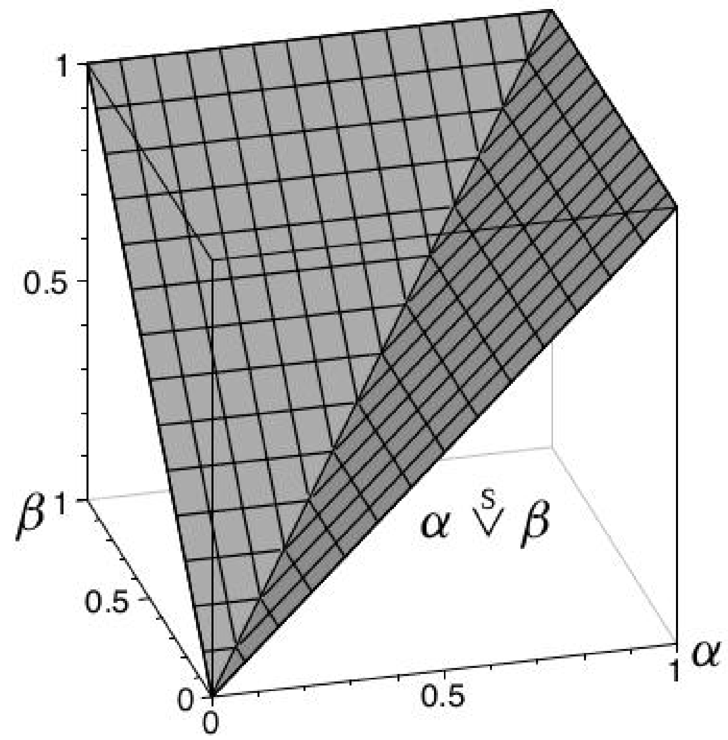
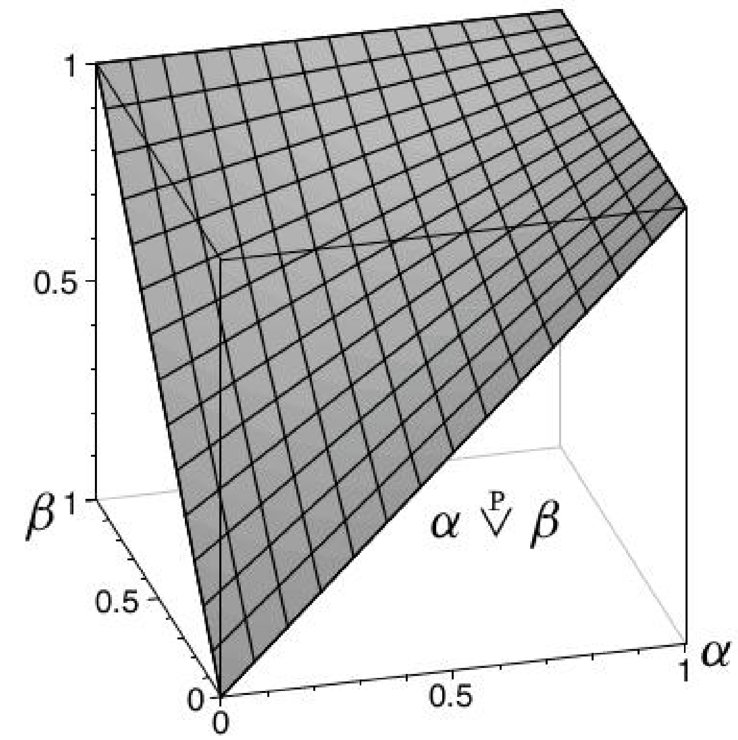
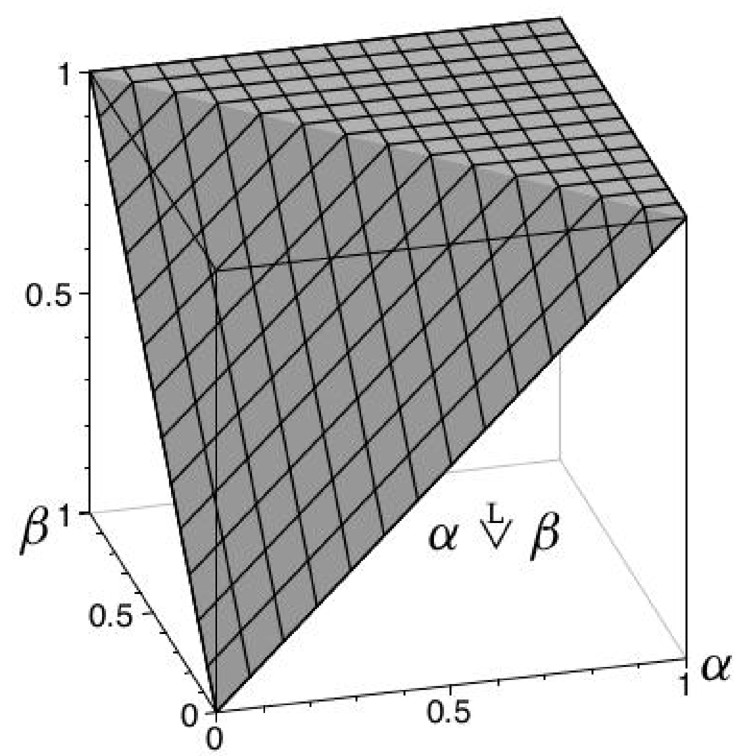
Basic fuzzy disjunctions

standard

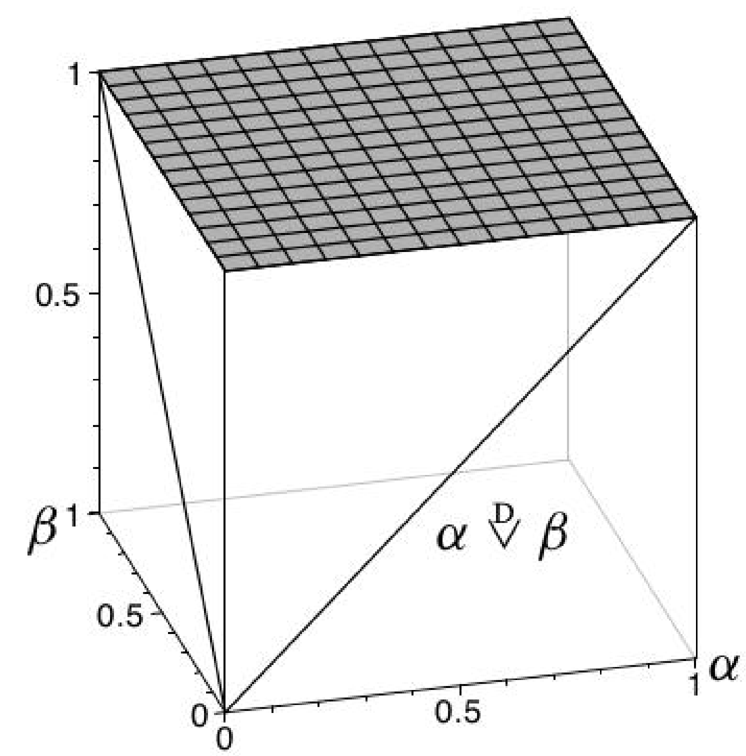


product





Łukasiewicz



drastic

- $$\alpha \overset{\mathrm{E}}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta}$$

Einstein fuzzy disjunction

- $$\alpha \overset{\mathrm{E}}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta}$$

Properties of fuzzy disjunctions

$$\forall \alpha, \beta \in [0,1] : \quad \alpha \overset{\mathrm{S}}{\vee} \beta \le \alpha \mathbin{\dot\vee} \beta \le \alpha \overset{\mathrm{D}}{\vee} \beta.$$

The standard disjunction is the only one which is idempotent, i.e., $\alpha \mathbin{\dot\vee} \alpha = \alpha$ for all $\alpha \in [0,1]$.

Duality

Let $\underset{\cdot}{\neg}$ be a fuzzy negation.

A. If $\underset{\cdot}{\wedge}$ is a fuzzy conjunction, then $\alpha \mathbin{\dot{\vee}} \beta = \underset{\cdot}{\neg}(\underset{\cdot}{\neg}\, \alpha \mathbin{\underset{\cdot}{\wedge}} \underset{\cdot}{\neg}\, \beta)$ is a fuzzy disjunction (**dual** to $\underset{\cdot}{\wedge}$ with respect to $\underset{\cdot}{\neg}$).

B. If $\dot{\vee}$ is a fuzzy disjunction, then $\alpha \mathbin{\underset{\cdot}{\wedge}} \beta = \underset{\cdot}{\neg}(\underset{\cdot}{\neg}\, \alpha \mathbin{\dot{\vee}} \underset{\cdot}{\neg}\, \beta)$ is a fuzzy conjunction (**dual** to $\dot{\vee}$ with respect to $\underset{\cdot}{\neg}$).

Proposition:

- The **Łukasiewicz** operations $\underset{\mathrm{L}}{\wedge}, \overset{\mathrm{L}}{\vee}$ are dual with respect to the **standard** negation.
- The **product** operations $\underset{\mathrm{P}}{\wedge}, \overset{\mathrm{P}}{\vee}$ are dual with respect to the **standard** negation.
- The **standard** operations $\underset{\mathrm{S}}{\wedge}, \overset{\mathrm{S}}{\vee}$ are dual with respect to **any** fuzzy negation.
- The **drastic** operations $\underset{\mathrm{D}}{\wedge}, \overset{\mathrm{D}}{\vee}$ are dual with respect to **any** fuzzy negation.

Classification of fuzzy disjunctions

A **continuous** fuzzy disjunction $\dot\vee$ is

- **Archimedean** if

$$\forall\alpha \in (0,1): \ \ \alpha \,\dot\vee\, \alpha > \alpha \qquad \text{(SA)}$$

- **strict** if

$$\forall\alpha \in [0,1)\ \forall\beta,\gamma \in [0,1]: \ \ \beta < \gamma \Rightarrow \alpha \,\dot\vee\, \beta < \alpha \,\dot\vee\, \gamma \qquad \text{(S3+)}$$

- **nilpotent** if it is Archimedean and not strict.

Equivalence of fuzzy disjunctions

Definition: We define a binary relation $\approx$ on fuzzy disjunctions such that

$$\overset{1}{\vee} \approx \overset{2}{\vee} \iff \exists i\colon [0,1] \to [0,1]\ \forall \alpha, \beta \in [0,1] : \alpha \overset{2}{\vee} \beta = i^{-1}\big(i(\alpha) \overset{1}{\vee} i(\beta)\big)\,.$$

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We denote by $[\dot{\vee}]$ the class of equivalence $\approx$ containing $\dot{\vee}$.

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Proposition: The set of all continuous (Archimedean, strict, nilpotent, resp.) fuzzy conjunctions is closed under $\approx$.

Representation theorems for strict fuzzy disjunctions

Theorem: An operation $\dot{\vee}\colon [0,1]^2 \to [0,1]$ is a **strict** fuzzy disjunction iff there is an increasing bijection $i\colon [0,1] \to [0,1]$ such that

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Proof: Take $j(\alpha) = \underset{\mathrm{S}}{\neg}\, i(\alpha)$, $\qquad j^{-1}(\gamma) = i^{-1}(\underset{\mathrm{S}}{\neg}\, \gamma)$,

$$\begin{aligned}
\alpha \,\dot\vee\, \beta &= i^{-1}\big(i(\alpha) \overset{\mathrm{P}}{\vee} i(\beta)\big) \\
&= \underbrace{i^{-1}\big(\underset{\mathrm{S}}{\neg}}_{j^{-1}}\big(\underbrace{\underset{\mathrm{S}}{\neg}\, i}_{j}\,(\alpha) \underset{\mathrm{P}}{\wedge} \underbrace{\underset{\mathrm{S}}{\neg}\, i}_{j}\,(\beta)\big)\big) \\
&= j^{-1}\big(j(\alpha) \cdot j(\beta)\big)\,.
\end{aligned}$$

Additive generators of fuzzy disjunctions

Theorem: An operation $\dot{\vee}\colon [0,1]^2 \to [0,1]$ is a **nilpotent** fuzzy disjunction iff there is an increasing bijection $i\colon [0,1] \to [0,1]$ (**additive generator**) such that

$$\alpha \mathbin{\dot{\vee}} \beta = i^{-1}\big(i(\alpha) \overset{\text{L}}{\vee} i(\beta)\big) = \begin{cases} i^{-1}\big(i(\alpha)+i(\beta)\big) & \text{if } i(\alpha)+i(\beta) \le 1\,, \\ 1 & \text{otherwise.} \end{cases}$$

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