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$A \wedge (A \Rightarrow B)$ $\alpha = e(A)$ β γ

$\alpha \dot\wedge (\alpha \dot\Rightarrow \beta) \overset{?}{=} \alpha \wedge_S \beta$

1. $\alpha \leq \beta$

$\alpha \dot\wedge 1 = \alpha$

2. $\beta < \alpha$

In part. $\wedge_P$: $\alpha \underset{P}{\Rightarrow} \beta = \frac{\beta}{\alpha}$ $\alpha \cdot \frac{\beta}{\alpha} = \beta$

$\dot\wedge$ strict - user generator

In general: $\alpha \dot\wedge \max\{\gamma : \alpha \dot\wedge \gamma \leq \beta\} = \beta$ $\gamma = \alpha \dot\Rightarrow \beta$

$\dot\wedge$ cont.

$\exists \gamma : \alpha \dot\wedge \gamma$

$\alpha \dot\wedge \gamma = \beta$

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$$((\alpha \underset{\cdot}{\to} \beta) \underset{\cdot}{\to} \beta) \wedge_S ((\beta \underset{\cdot}{\to} \alpha) \underset{\cdot}{\to} \alpha) \overset{?}{=} \alpha \vee_S \beta$$

WLOG: $\alpha \leq \beta$

$$\underbrace{\underbrace{(\alpha \underset{\cdot}{\to} \beta)}_{1} \underset{\cdot}{\to} \beta}_{\beta} \wedge_S \underbrace{((\beta \underset{\cdot}{\to} \alpha) \underset{\cdot}{\to} \alpha)}_{\geq \beta}$$

$$\beta \underset{\cdot}{\to} \alpha \geq 1 \underset{\cdot}{\to} \alpha = \alpha$$

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PRODUCT LOGIC

Standard semantics:

$\wedge$ ... product (or any strict) fuzzy conjunction $\underset{P}{\wedge} = \cdot$

$\rightarrow$ ... residuum of $\underset{P}{\wedge}$, Goguen implication $\underset{P}{\rightarrow}$

$\neg$ ... Gödel negation $\underset{G}{\neg}$

There is no connective in product logic which is interpreted by the standard fuzzy negation.

Syntax:

Axioms (A1)–(A7) and

(P1) $\quad \neg\neg C \rightarrow ((A \wedge C \rightarrow B \wedge C) \rightarrow (A \rightarrow B))$

(P2) $\quad A \wedge (A \rightarrow \neg A) \rightarrow \mathbf{0}$

Remark: $e(\neg\neg C) = 1$ iff $e(C) \neq 0$ and

$\neg\neg C \vdash (A \wedge C \rightarrow B \wedge C) \leftrightarrow (A \rightarrow B)$

Alternatively, we may use a single axiom [Cintula] instead of (P1),(P2):

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Deduction rule: Modus Ponens

Standard completeness of product logic

$(\vdash A) \iff (\models A)$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0,1]$ with product operations).

Moreover, for any finite theory $\mathcal{T}$, $(\mathcal{T} \vdash A) \iff (\mathcal{T} \models A)$.

ŁUKASIEWICZ LOGIC

Standard semantics:

$\wedge$ ... Łukasiewicz (or any nilpotent) fuzzy conjunction $\underset{\mathsf{L}}{\wedge}$

$\to$ ... residuum of $\underset{\mathsf{L}}{\wedge}$, Łukasiewicz implication $\underset{\mathsf{L}}{\to}$

$\neg$ ... standard fuzzy negation $\underset{\mathsf{S}}{\neg}\, x = 1 - x$

Syntax:

Axioms (A1)–(A7) and

(L) $\qquad \neg\neg A \to A$

Corollary: $\qquad \neg\neg A \leftrightarrow A$

Deduction rule: Modus Ponens

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Deduction rule: Modus Ponens

Only here the negation is interpreted by the standard negation, $e(\neg A) = \underset{\mathsf{S}}{\neg}\, e(A)$

Example L1: $(B \to A) \leftrightarrow (\neg A \to \neg B)$

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(A1): $(A \to B) \to ((B \to C) \to (A \to C))$

$(A \to B) \to (\neg B \to \neg A)$

$A := \neg A$ $\quad$ $B := \neg B$

$(\neg A \to \neg B) \to (\neg\neg B \to \neg\neg A)$

Example L1: $(B \to A) \leftrightarrow (\neg A \to \neg B)$

$$\begin{aligned} \text{(A1)}: &\quad (A \to B) \to ((B \to C) \to (A \to C)) \\ (C := 0): &\quad (A \to B) \to (\neg B \to \neg A) \\ (A := \neg A, B := \neg B): &\quad \neg(A \to \neg B) \to (\underbrace{\neg\neg B}_{B} \to \underbrace{\neg\neg A}_{A}) \end{aligned}$$

Example L2:

$$(A \to B) \leftrightarrow (\neg(A \wedge \neg B)) \tag{1}$$

Example L2:

$$(A \to B) \leftrightarrow (\neg(A \land \neg B)) \tag{1}$$

"1 ← 1'":

$$A \to (\neg B \to A \land \neg B)$$

$$A \to (\neg(A \land \neg B) \to \neg\neg B)$$

$$\neg(A \land \neg B) \to (A \to B)$$

Example L2:

$$(A \to B) \leftrightarrow (\neg(A \wedge \neg B)) \tag{1}$$

"$\to$":

(A1) $$\underbrace{(1 \to A)} \to ((A \to B) \to \underbrace{(1 \to B)})$$

$$A \to \underbrace{((A \to B) \to B)}$$

(Example L1) : $$A \to \overbrace{(\neg B \to \neg(A \to B))}$$

(A5a) : $$A \wedge \neg B \to \neg(A \to B)$$

(Example L1) : $$(A \to B) \to \neg(A \wedge \neg B)$$

Example L2:

$$(A \to B) \leftrightarrow (\neg(A \wedge \neg B)) \tag{1}$$

"$\rightarrow$":

$$\begin{aligned}
\text{(A1)} \quad & \underbrace{(1 \to A)} \to ((A \to B) \to \underbrace{(1 \to B)}) \\
& A \to \underbrace{((A \to B) \to B)} \\
\text{(Example L1)}: \quad & A \to \overbrace{(\neg B \to \neg(A \to B))} \\
\text{(A5a)}: \quad & A \wedge \neg B \to \neg(A \to B) \\
\text{(Example L1)}: \quad & (A \to B) \to \neg(A \wedge \neg B)
\end{aligned}$$

"$\leftarrow$":

$$\begin{aligned}
\text{(Example 4, } B := \neg B\text{)}: \quad & A \to \underbrace{(\neg B \to (A \wedge \neg B))} \\
\text{(Example L1)}: \quad & A \to \overbrace{(\neg(A \wedge \neg B) \to B)} \\
\text{(Exchange axiom)}: \quad & \neg(A \wedge \neg B) \to (A \to B)
\end{aligned}$$

Corollary:

$$A \wedge B = \neg(A \to \neg B) \tag{2}$$

Corollary:

$$A \wedge B = \neg(A \rightarrow \neg B) \tag{2}$$

$$\neg(A \wedge \neg B) = A \rightarrow B$$

$$B := \neg B$$

$$\neg(A \wedge B) = \neg(A \rightarrow \neg B)$$

Corollary:

$$A \wedge B = \neg(A \to \neg B) \tag{2}$$

Alternative axiomatization [Łukasiewicz & Tarski] with only $\to$, $\mathbf{0}$, conjunction is considered a connective **derived** by (2),

(L1) $A \to (B \to A)$

(L2) $(A \to B) \to \big((B \to C) \to (A \to C)\big)$

(L3) $(\neg A \to \neg B) \to (B \to A)$

(L4) $\big((A \to B) \to B\big) \to \big((B \to A) \to A\big)$ (**Łukasiewicz axiom**)

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Verification of these axioms in (BL) with (L):

(L1)=(C1) (valid in BL, see Example 1)

(L2)=(A1)

(L3)=(C3, see Example L1)

Corollary:

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(L4)

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$$\neg A \wedge (\neg A \to \neg B) \to \neg B \wedge (\neg B \to \neg A)$$

$$\neg A \wedge (B \to A) \to \neg B \wedge (A \to B)$$

$$(B \to A) \wedge \neg A \to (A \to B) \wedge \neg B$$

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$$((A \to B) \to B) \to ((B \to A) \to A)$$

(L4)

$$
\begin{array}{rr}
(\mathsf{A4}, A := \neg A, B := \neg B) & \neg A \wedge \underbrace{(\neg A \to \neg B)} \to \neg B \wedge \underbrace{(\neg B \to \neg A)} \\
(\text{Example L1}) & \underbrace{\neg A \wedge \overbrace{(B \to A)}} \to \underbrace{\neg B \wedge \overbrace{(B \to A)}} \\
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(\text{Example L1}) & \neg\big((B \to A) \wedge \neg A\big) \to \neg\big((B \to A) \wedge \neg B\big) \\
(2) & \big((B \to A) \to A\big) \to \big((B \to A) \to B\big)
\end{array}
$$

$$A \overset{\mathsf{S}}{\vee} B = \Big((A \to B) \to B\Big) \underset{\mathsf{S}}{\wedge} \Big((B \to A) \to A\Big)$$

$$\text{(L4)} \quad \Big((A \to B) \to B\Big) \to \Big((B \to A) \to A\Big) \qquad \text{(Łukasiewicz axiom)}$$

$$\Big((A \to B) \to B\Big) \leftrightarrow \Big((B \to A) \to A\Big)$$

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(L4) $$\big((A \to B) \to B\big) \to \big((B \to A) \to A\big)$$ (Łukasiewicz axiom)

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RATIONAL PAVELKA LOGIC (RPL)

[Pavelka] Instead of $\{A_1, \ldots, A_k\} \vdash D_n$:

$$D_1$$
$$D_2$$
$$\vdots$$
$$D_n$$

we want graded formulas like (A_i, r_i) ($r_i \in [0,1] \cap \mathbb{Q}$) ($\mathbb{Q}$ for countability):
$\{(A_1, r_1), \ldots, (A_k, r_k)\} \vdash (D_n, s_n)$:

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Graded formulas are proved using **graded axioms** and **graded deduction rules** like
Generalized Modus Ponens GMP : $\dfrac{(A, r),\ (A \to B, s)}{(B, r \underset{\text{Ł}}{\wedge} s)}$

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HÁJEK'S FORMULATION OF RATIONAL PAVELKA LOGIC

RPL as a minimal modification of Łukasiewicz logic:
Add constants $\mathbf{r}$, $r \in [0,1] \cap \mathbb{Q}$ (where $\mathbf{0}$, $\mathbf{1} = \neg\mathbf{0}$ keep the previous meaning)

Syntax:
Axioms of Łukasiewicz logic ((A1)–(A7) and (L), or (L1)–(L4)),
and **bookkeeping axioms**:
(RPL) $\quad (\mathbf{r} \to \mathbf{s}) \leftrightarrow \mathbf{t} \qquad \forall r, s, t \in [0,1] \cap \mathbb{Q} : \; r \underset{\mathrm{L}}{\to} s = t$

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Deduction rule: Modus Ponens

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$$\frac{(A, r), \ (A \to B, s)}{(B, \underbrace{r \wedge_L s}_{t})}$$

$$\frac{r \to A, \quad s \to (A \to B)}{t \to B}$$

$$t = r \wedge_L s$$

Deduction rule: Modus Ponens

Generalized Modus Ponens is obtained as a special case of Modus Ponens of Łukasiewicz logic:

$$\frac{\mathbf{r} \to A,\ \mathbf{s} \to (A \to B)}{\mathbf{t} \to B} \qquad \forall r, s, t \in [0, 1] \cap \mathbb{Q} :\ t = r \underset{\text{L}}{\wedge} s$$

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$$\begin{aligned}
\text{SA} : &\quad D_1 = \mathbf{r} \to A \\
\text{SA} : &\quad D_2 = \mathbf{s} \to (A \to B) \\
\text{Exchange rule for } D_2 : &\quad D_3 = A \to (\mathbf{s} \to B) \\
\text{TI}(D_1, D_3) : &\quad D_4 = \mathbf{r} \to (\mathbf{s} \to B) \\
\text{(A5a)}, A := \mathbf{r}, B := \mathbf{s}, C := B : &\quad D_5 = (\mathbf{r} \to (\mathbf{s} \to B)) \to (\mathbf{r} \wedge \mathbf{s} \to B) \\
\text{MP}(D_4, D_5) : &\quad D_6 = \mathbf{r} \wedge \mathbf{s} \to B \\
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The deduction theorem of Łukasiewicz logic remains valid in RPL.

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Semantics:

$e(\mathbf{r}) = r \qquad \forall r \in [0,1] \cap \mathbb{Q}$

Thus $e(\mathbf{r} \to A) = 1 \iff e(A) \geq r$

Besides theorems and 1-tautologies (defined as usual), for a formula A and theory $\mathcal{T}$, we define

- the **truth degree** $||A||_{\mathcal{T}} = \inf\{e(A) \mid \forall B \in \mathcal{T} : e(B) = 1\}$
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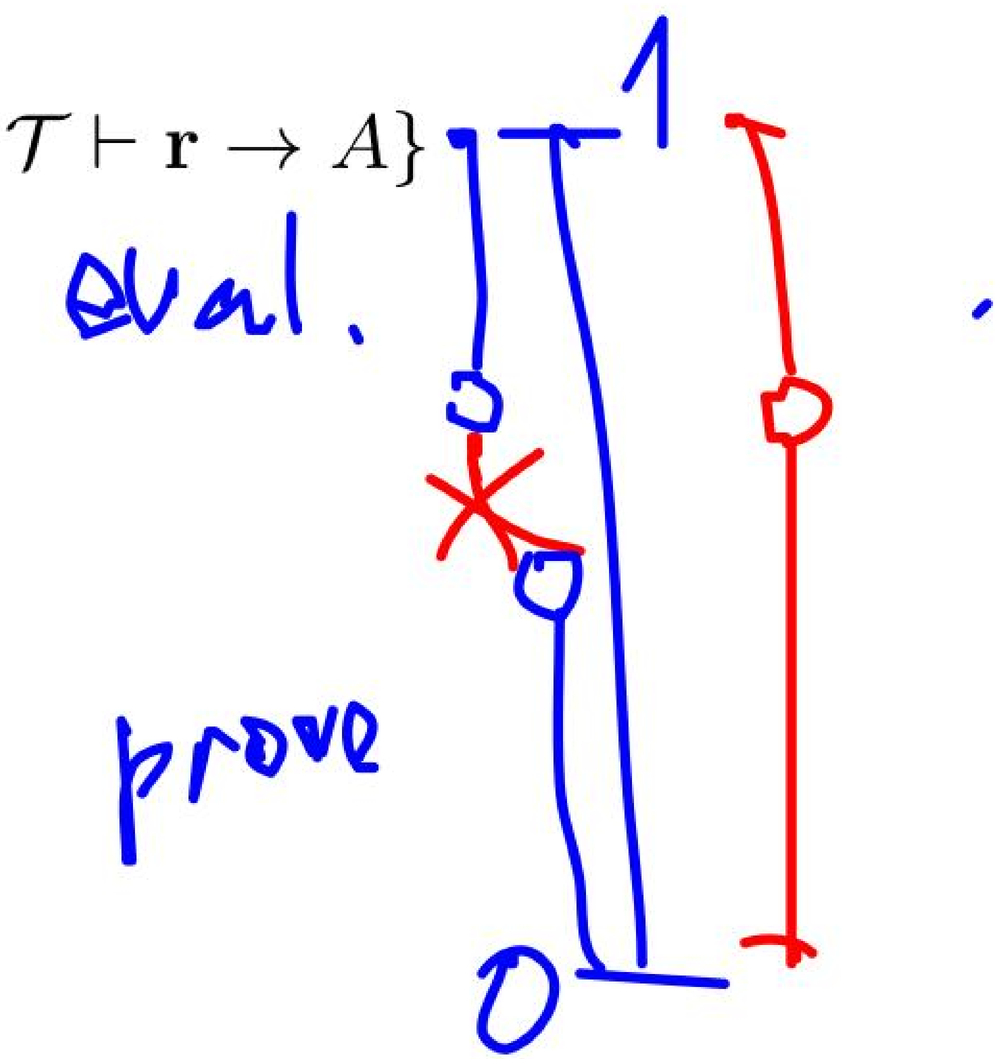


Completeness Theorem for RPL: $|A|_{\mathcal{T}} = ||A||_{\mathcal{T}}$

Theory $\mathcal{T}$ is

- **consistent**, if $\mathcal{T} \nvdash \mathbf{0}$,
  equivalently, if $\mathcal{T} \nvdash \mathbf{r} \qquad \forall r < 1$;

- **complete**, if $\mathcal{T} \vdash A \to \mathbf{r}$ or $\mathcal{T} \vdash \mathbf{r} \to A \qquad \forall A\ \forall r \in [0,1] \cap \mathbb{Q}$.

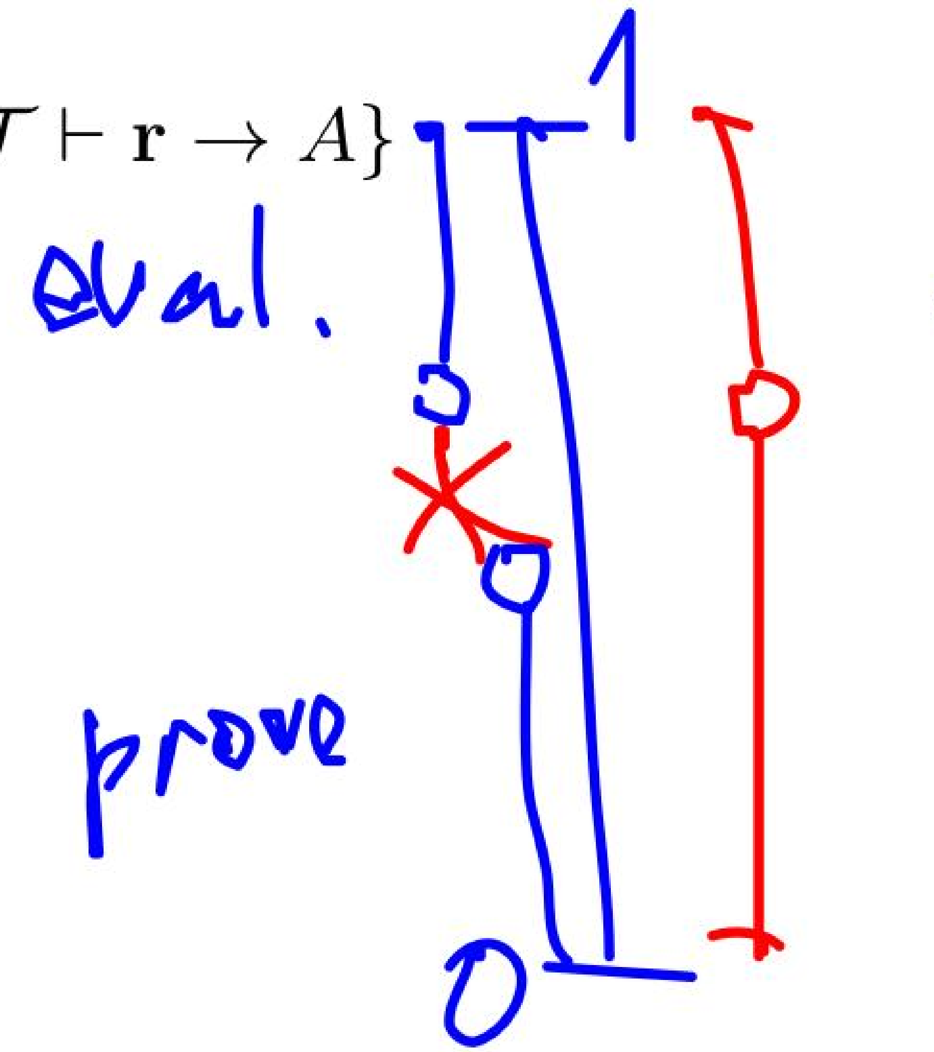
Lemma: Theory $\mathcal{T}$ is inconsistent iff $\mathcal{T} \vdash A \qquad \forall A$.

Lemma: If $(\mathcal{T} \nvdash \mathbf{r} \to A)$, then $\mathcal{T} \cup \{A \to \mathbf{r}\}$ is consistent.

Besides theorems and 1-tautologies (defined as usual), for a formula A and theory $\mathcal{T}$, we define

- the **truth degree** $||A||_{\mathcal{T}} = \inf\{e(A) \mid \forall B \in \mathcal{T} : e(B) = 1\}$

- the **provability degree** $|A|_{\mathcal{T}} = \sup\{r \in [0,1] \cap \mathbb{Q} \mid \mathcal{T} \vdash \mathbf{r} \to A\}$



Completeness Theorem for RPL: $|A|_{\mathcal{T}} = ||A||_{\mathcal{T}}$

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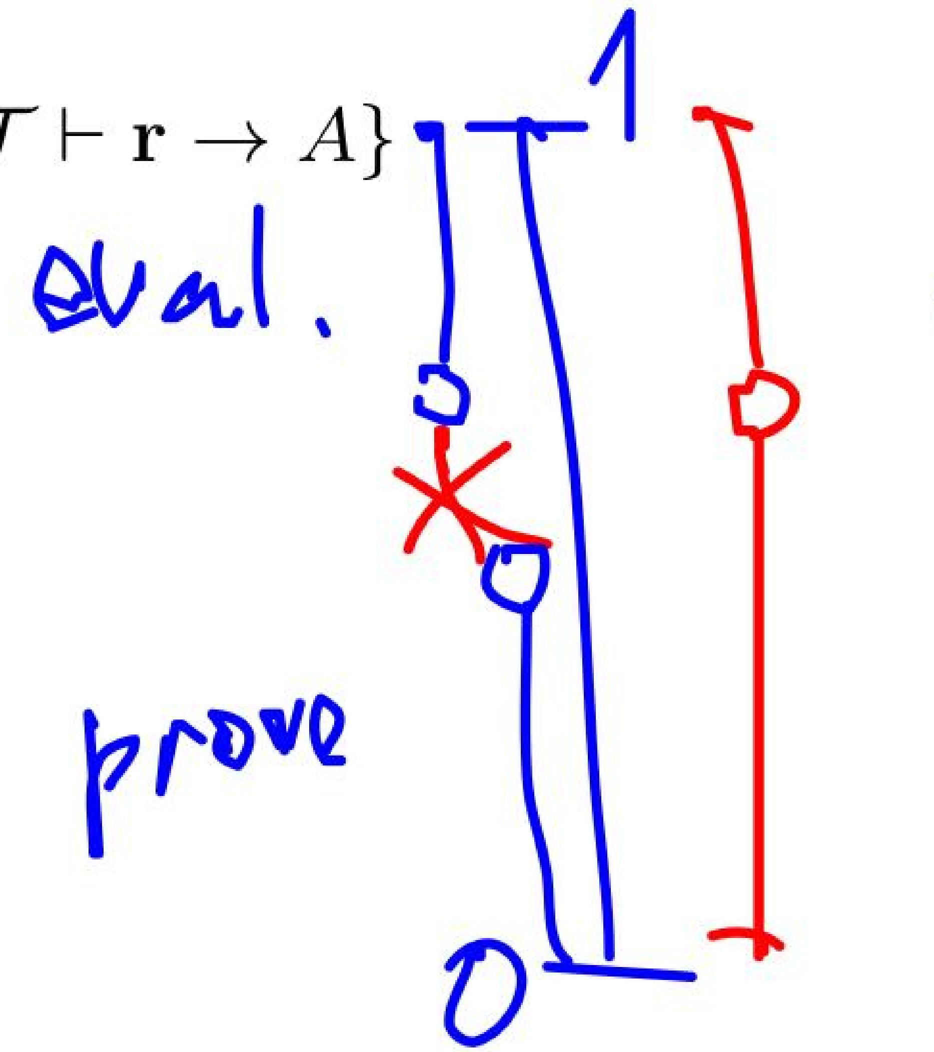

<u>**Completeness Theorem for RPL:**</u> $|A|_{\mathcal{T}} = ||A||_{\mathcal{T}}$

Theory $\mathcal{T}$ is

- **consistent**, if $\mathcal{T} \not\vdash \mathbf{0}$,
  equivalently, if $\mathcal{T} \not\vdash \mathbf{r} \qquad \forall r < 1$;

- **complete**, if $\mathcal{T} \vdash A \to \mathbf{r}$ or $\mathcal{T} \vdash \mathbf{r} \to A \qquad \forall A\ \forall r \in [0,1] \cap \mathbb{Q}$.

Lemma: Theory $\mathcal{T}$ is inconsistent iff $\mathcal{T} \vdash A \qquad \forall A$.

Lemma: If $(\mathcal{T} \not\vdash \mathbf{r} \to A)$, then $\mathcal{T} \cup \{A \to \mathbf{r}\}$ is consistent.

Lemma: Let a theory $\mathcal{T}$ be consistent and complete. Then

- $|A|_{\mathcal{T}} = \sup\{r \in [0,1] : \mathcal{T} \vdash \mathbf{r} \to A\} = \inf\{s \in [0,1] : \mathcal{T} \vdash A \to \mathbf{s}\}$;

- $|\cdot|_{\mathcal{T}}$ comutes with logical connectives, i.e., $|A \to B|_{\mathcal{T}} = |A|_{\mathcal{T}} \underset{\mathsf{L}}{\to} |B|_{\mathcal{T}}$ (in particular, $|\neg A|_{\mathcal{T}} = 1 - |A|_{\mathcal{T}}$);

- function e defined by $e(A) = |A|_{\mathcal{T}}$ is an evaluation.

COMPACTNESS OF LOGICS

Compactness Theorem I: A theory is satisfiable iff each its **finite** subset (*subtheory*) is satisfiable.

Compactness Theorem I holds in classical, basic, Gödel, Łukasiewicz, and product logic and in RPL.

Lemma: Let a theory $\mathcal{T}$ be consistent and complete. Then

- $|A|_{\mathcal{T}} = \sup\{r \in [0,1] : \mathcal{T} \vdash \mathbf{r} \to A\} = \inf\{s \in [0,1] : \mathcal{T} \vdash A \to \mathbf{s}\}$;

- $|\cdot|_{\mathcal{T}}$ comutes with logical connectives, i.e., $|A \to B|_{\mathcal{T}} = |A|_{\mathcal{T}} \underset{\text{Ł}}{\to} |B|_{\mathcal{T}}$ (in particular, $|\neg A|_{\mathcal{T}} = 1 - |A|_{\mathcal{T}}$);

- function e defined by $e(A) = |A|_{\mathcal{T}}$ is an evaluation.