Regular measures on tribes of fuzzy sets

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Related work presented at Linz Seminars

1979

Henri M. Prade: Nomenclature of fuzzy measures

Erich Peter Klement: Extension of probability measures to fuzzy measures and their characterization

Werner Schwyhla: Conditions for a fuzzy probability measure to be an integral

Josette and Jean-Louis Coulon: Fuzzy boolean algebras

1980

Erich Peter Klement: Some remarks on t-norms, fuzzy σ-algebras and fuzzy measures

Werner Schwyhla: Remarks on non-additive measures and fuzzy sets

Ulrich Höhle: Fuzzy measures as extensions

1981

Erich Peter Klement: Fuzzy measures assuming their values in the set of fuzzy numbers

1982

Erich Peter Klement: On different approaches to fuzzy probabilities

Didier Dubois: Upper and lower possibilistic expectations and applications

Ronald R. Yager: Probabilities from fuzzy observations

1983

Siegfried Weber: How to measure fuzzy sets

1984

Robert Lowen: Spaces of probability measures revisited

1985

Siegfried Weber: Generalizing the axioms of probability

1986

Erich Peter Klement: Representation of crisp- and fuzzy-valued measures by integrals

Siegfried Weber: Some remarks on the theory of pseudo-additive measures and its applications

1987

Erich Peter Klement: On a class of non-additive measures and integrals

1988

Alain Chateauneuf: Decomposable measures, distorted probabities and concave capacities

Siegfried Weber: Decomposable measures for conditional objects

Aldo Ventre: A Yosida-Hewitt like theorem for $\perp$-decomposable measures (joint paper with M. Squillante)

Massimo Squillante: $\perp$-decomposable measures and integrals: Convergence and absolute continuity (joint paper with L. D'Apuzzo and R. Sarno)

Ulrich Höhle: Non-classical models of probability theory

⋮

1998

Mirko Navara, Pavel Pták: Types of uncertainty and the role of the Frank t-norms in classical and nonclassical logics

Mirko Navara: Nearly Frank t-norms and the characterization of T-measures

Giuseppina Barbieri: A representation theorem and a Liapounoff theorem for T_s-measures

Beloslav Riečan: On the probability theory and fuzzy sets

Ulrich Höhle: Realizations for generalized probability measures

Marc Roubens: On probabilistic interactions among players in cooperative games

Radko Mesiar: k-order pseudo-additive measures

$\vdots$

Classical measure theory [Halmos]

THEOREMS about

FUNCTIONALS (MEASURES) on

SETS

Also [Sugeno; Dubois, Prade; Wang, Klir; Pap]

What is fuzzy measure theory?

THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

SETS

[Feng; Guo, Zhang, Wu]

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[Butnariu, Klement]

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[Butnariu, Klement, Mesiar, Barbieri, Weber]

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[Butnariu, Klement, Mesiar, Barbieri, Weber]

Also measure theory on MV-algebras [Cignoli, D'Ottaviano, Mundici, Riečan]

Basic fuzzy logical operations

Standard negation, $\neg x = 1 - x$

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Fuzzy disjunction (t-conorm): $S\colon [0,1]^2 \to [0,1]$ dual to T:

$$S(x,y) = \neg T(\neg x, \neg y)$$

Basic notions of fuzzy measure theory

classical measure theory	**fuzzy** measure theory
σ-**algebra** $\mathcal{T} \subseteq \mathbf{2}^X$	**tribe** $(\mathcal{T}, T)$, where $\mathcal{T} \subseteq [\mathbf{0}, \mathbf{1}]^X$
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Always: Crisp elements of $\mathcal{T}$, i.e., $\mathcal{T} \cap \{0,1\}^X$, determine a σ-algebra $\mathcal{B}$

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Full tribes

Example: Let $\mathcal{B}$ be a σ-algebra of subsets of X,
$\mathcal{T}$ be the corresponding collection of characteristic functions (indicators):

$$\mathcal{T} = \{\chi_A \mid A \in \mathcal{B}\} .$$

Then $(\mathcal{T}, T)$ is a tribe (for any t-norm T).
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Łukasiewicz t-norm

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$$

These tribes correspond to set-representable σ-complete MV-algebras

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