SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1)	$(A \to B) \to ((B \to C) \to (A \to C))$
(A2)	$A \wedge B \to A$
(A3)	$A \wedge B \to B \wedge A$
(A4)	$A \wedge (A \to B) \to B \wedge (B \to A)$
(A5a)	$(A \to (B \to C)) \to (A \wedge B \to C)$
(A5b)	$(A \wedge B \to C) \to (A \to (B \to C))$
(A6)	$((A \to B) \to C) \to (((B \to A) \to C) \to C)$
(A7)	$\mathbf{0} \to A$

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Notation: $\vdash A$, $\mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

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Notation: $\vdash A$, $\mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

$$\begin{aligned} \text{(A2)}: &\quad D_1 = A \wedge B \to A \\ \text{(A5b)}, C := A: &\quad D_2 = (A \wedge B \to A) \to (A \to (B \to A)) \\ \text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A) \end{aligned}$$

$\Rightarrow$ (C1) can be added to axioms of BL

Proposition 1 Consequence of (A1):

$$\{A \to B,\ B \to C\} \vdash A \to C$$

$\Rightarrow$ we can add a deduction rule

TI($A \to B,\ B \to C$): $\dfrac{A \to B,\ B \to C}{A \to C}$ (transitivity of implication)

Example 2 $\vdash (A \to (B \to C)) \to (B \to (A \to C))$

(**Exchange rule**, also called "exchange axiom")

Notation: $\vdash A$, $\mathcal{T} \vdash A$

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(A7)	$\mathbf{0} \to A$

Notation: $\vdash A$, $\mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

Notation: $\vdash A$, $\mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

(A2) $A \wedge B \to A$

SYNTAX OF BASIC LOGIC 2

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(A3) $A \wedge B \to B \wedge A$

(A4) $A \wedge (A \to B) \to B \wedge (B \to A)$ $\leftrightarrow (A \wedge_S B)$

(A5a) $(A \to (B \to C)) \to (A \wedge B \to C)$

(A5b) $(A \wedge B \to C) \to (A \to (B \to C))$

(A6) $((A \to B) \to C) \to (((B \to A) \to C) \to C)$

(A7) $\mathbf{0} \to A$

Notation: $\vdash A,\qquad \mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

(A2) $A \wedge B \to A$

Notation: $\vdash A$, $\mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

(A2) $A \wedge B \to A$

(A5b) $(A \wedge B \to A) \to (A \to (B \to A))$

MP

$A \vdash B \to A$

$0 \to B$ $B \to 1$

Notation: $\vdash A$, $\mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

$$\begin{aligned} \text{(A2)}: &\quad D_1 = A \wedge B \to A \\ \text{(A5b)}, C := A: &\quad D_2 = (A \wedge B \to A) \to (A \to (B \to A)) \\ \text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A) \end{aligned}$$

$\Rightarrow$ (C1) can be added to axioms of BL

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$$\{A \to B,\ B \to C\} \vdash A \to C$$

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(**Exchange rule**, also called "exchange axiom")

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(**Exchange rule**, also called "exchange axiom")

(A1) $(B \wedge A \to A \wedge B) \to ((A \wedge B \to C) \to (B \wedge A \to C))$

(A3) $B \wedge A \to A \wedge B$

MP: $(A \wedge B \to C)_2 \to (B \wedge A \to C)_3$

(A5a): $(A \to (B \to C))_1 \to (A \wedge B \to C)_2$

(A5b): $(B \wedge A \to C)_3 \to (B \to (A \to C))_4$

$X, Y \vdash X \wedge Y$

$A \to B, B \to A \vdash A \leftrightarrow B =$

$= (A \to B) \wedge (B \to A)$

Example 2 $\vdash (A \to (B \to C)) \to (B \to (A \to C))$

(**Exchange rule**, also called "exchange axiom")

$$
\begin{aligned}
\text{(A1) } A := B \wedge A, & \\
B := A \wedge B: \quad & D_1 = (B \wedge A \to A \wedge B) \to ((A \wedge B \to C) \to (B \wedge A \to C)) \\
\text{(A3) } A :=: B: \quad & D_2 = B \wedge A \to A \wedge B \\
\text{MP}(D_2, D_3): \quad & D_3 = (A \wedge B \to C) \to (B \wedge A \to C) \\
\text{(A5a)}: \quad & D_4 = (A \to (B \to C)) \to (A \wedge B \to C) \\
\text{(A5b) } A :=: B: \quad & D_5 = (B \wedge A \to C) \to (B \to (A \to C)) \\
\text{TI}(D_4, D_3): \quad & D_6 = (A \to (B \to C)) \to (B \wedge A \to C) \\
\text{TI}(D_6, D_5): \quad & D_7 = (A \to (B \to C)) \to (B \to (A \to C))
\end{aligned}
$$

Example 3 $\vdash A \to A$

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Exch. axiom (an axiom)

$$(A \to (B \to A)) \to (B \to (A \to A))$$

(c1) proved

$$A \to (B \to A)$$

MP

$$B \to (A \to A)$$

MP

$$A \to A$$

Example 3 $\quad \vdash A \to A$

For brevity, let B denote a provable formula, e.g., axiom (A1).

$$
\begin{aligned}
\text{(A1)}: &\quad D_1 = B \\
\text{Ex. 2}, C := A: &\quad D_2 = (A \to (B \to A)) \to (B \to (A \to A)) \\
\text{(C1)}: &\quad D_3 = A \to (B \to A) \\
\text{MP}(D_3, D_2): &\quad D_4 = B \to (A \to A) \\
\text{MP}(D_1, D_4): &\quad D_5 = A \to A
\end{aligned}
$$

Deduction theorem in basic logic

$\mathcal{T}$ ... theory

A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\exists n \in N : (\mathcal{T} \vdash A^n \to B)$,

where $A^n = \underbrace{(A \wedge (A \wedge \cdots (A \wedge A) \cdots))}_{n\times}$

In view of (A5a), (A5b),

$(A^n \to B) \leftrightarrow \underbrace{(A \to (A \to \cdots (A \to}_{n\times} B) \cdots))$

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INTERPLAY OF SYNTAX AND SEMANTICS OF BASIC LOGIC

Soundness Each provable formula is a 1-tautology, i.e., if $\vdash A$, then $\models A$. Moreover, for any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness Each 1-tautology is provable, i.e., if $\models A$, then $\vdash A$.

Strong completeness [Hájek 1998]
For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$.
(We consider all evaluations with values in BL-algebras.)

Standard completeness [Cignoli, R., Esteva, F., Godo, L., Torrens, A. 2000]
Each formula which is evaluated to 1 by all **standard** evaluations (with values in $[0, 1]$ and an arbitrary continuous fuzzy conjunction) is provable.

Exercise Which axioms of the classical logic are 1-tautologies of BL (and hence provable in BL)?

Exchange rule allows us to rewrite (A1):
$(B \to C) \to ((A \to B) \to (A \to C))$

Deduction theorem in basic logic

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$\mathcal{T} \cup \{A\} \vdash B$ iff $\exists n \in N : (\mathcal{T} \vdash A^n \rightarrow B)$,
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In view of (A5a), (A5b),
$(A^n \rightarrow B) \leftrightarrow \underbrace{(A \rightarrow (A \rightarrow \cdots (A \rightarrow}_{n\times} B) \cdots))$

Deduction theorem in basic logic

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$\mathcal{T} \cup \{A\} \vdash B$ iff $\exists n \in N : (\mathcal{T} \vdash A^n \to B)$,
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In view of (A5a), (A5b),
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Exchange rule allows us to rewrite (A1):
$(B \to C) \to ((A \to B) \to (A \to C))$ $\leftrightarrow ((A \to B) \to ((B \to C) \to (A \to C)))$

Example 4 $\vdash A \to (B \to A \land B)$

Example 4 $\vdash A \to (B \to A \wedge B)$

$$A \wedge B \to A \wedge B$$

(A5b) $((A \wedge B \to A \wedge B) \to (A \to (B \to (A \wedge B))))$

MP: $A \to (B \to (A \wedge B))$

Example 4 $\vdash A \to (B \to A \wedge B)$

$$\begin{aligned}
\text{Ex. 3}: \quad & D_1 = A \wedge B \to A \wedge B \\
\text{(A5b)}, C := A \wedge B: \quad & D_2 = (A \wedge B \to A \wedge B) \to (A \to (B \to A \wedge B)) \\
\text{MP}(D_1, D_2): \quad & D_3 = A \to (B \to A \wedge B)
\end{aligned}$$

Corollary $A \vdash B \to A \wedge B$

$\{A, B\} \vdash A \wedge B$

Example 4 $\vdash A \to (B \to A \wedge B)$

$A \wedge B \to A \wedge B$

(A5b) $(A \wedge B \to A \wedge B) \to (A \to (B \to (A \wedge B)))$

MP: $A \to (B \to (A \wedge B))$

Example 4 $\vdash A \to (B \to A \land B)$

$$A \land B \to A \land B$$

$$(A5b)\quad \big((A \land B \to A \land B) \to (A \to (B \to (A \land B)))\big)$$

$$\text{MP:}\quad A \to (B \to (A \land B))$$

Example 4 $\quad \vdash A \to (B \to A \wedge B)$

$$\begin{aligned} \text{Ex. 3}: &\quad D_1 = A \wedge B \to A \wedge B \\ \text{(A5b)}, C := A \wedge B: &\quad D_2 = (A \wedge B \to A \wedge B) \to (A \to (B \to A \wedge B)) \\ \text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A \wedge B) \end{aligned}$$

Corollary $\quad A \vdash B \to A \wedge B$

$\{A, B\} \vdash A \wedge B$

Example 4 $\vdash A \to (B \to A \wedge B)$

$$\begin{aligned}
\text{Ex. 3}: &\quad D_1 = A \wedge B \to A \wedge B \\
\text{(A5b)}, C := A \wedge B: &\quad D_2 = (A \wedge B \to A \wedge B) \to (A \to (B \to A \wedge B)) \\
\text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A \wedge B)
\end{aligned}$$

Corollary $A \vdash B \to A \wedge B$

$\{A, B\} \vdash A \wedge B$

$X \to Y, Y \to X \vdash (X \to Y) \wedge (Y \to X)$

$= X \Leftrightarrow Y$

Example of deduction

$$A \vdash B \to A \wedge (A \wedge B)$$

Example of deduction

$A \vdash B \to A \wedge (A \wedge B)$

A

(Ax5b) $A \to (A \wedge B \to A \wedge (A \wedge B))$

MP: $A \wedge B \to A \wedge (A \wedge B)$

(A5b) $(A \wedge B \to A \wedge (A \wedge B)) \to (A \to (B \to A \wedge (A \wedge B)))$

MP: $A \to (B \to A \wedge (A \wedge B))$

MP: $B \to A \wedge (A \wedge B)$

Example of deduction $\quad A \vdash B \to A \wedge (A \wedge B)$

$$\begin{aligned}
\text{SA}: &\quad D_1 = A \\
\text{Ex. 4, } B := A \wedge B: &\quad D_2 = A \to (A \wedge B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_2): &\quad D_3 = A \wedge B \to A \wedge (A \wedge B) \\
\text{(A5b)}, & \\
C := A \wedge (A \wedge B): &\quad D_4 = (A \wedge B \to A \wedge (A \wedge B)) \to (A \to (B \to A \wedge (A \wedge B))) \\
\text{MP}(D_3, D_4): &\quad D_5 = A \to (B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_5): &\quad D_6 = B \to A \wedge (A \wedge B)
\end{aligned}$$

Here $\nvdash A \to (B \to A \wedge (A \wedge B))$
but $\vdash A \wedge A \to (B \to A \wedge (A \wedge B))$
$\quad \vdash A \to (A \to (B \to A \wedge (A \wedge B)))$
$\quad \vdash A \to (A \wedge B \to A \wedge (A \wedge B))$
(by substitution $B := A \wedge B$ in Example 4)

Exercise $\quad$ Prove in BL: $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

1. How do the properties of interpretation of conjunction (fuzzy conjunction) follow from logical axioms?
2. How can we simplify our proofs?

Corollary of (A1)**:**

$B \leftrightarrow C \vdash (A \to B) \leftrightarrow (A \to C)$

$B \leftrightarrow C \vdash (B \to A) \leftrightarrow (C \to A)$

(We are not yet allowed to work similarly with the conjunction.)

Commutativity of $\underset{\cdot}{\wedge}$ follows directly from (A3).

Example of deduction $\quad A \vdash B \to A \wedge (A \wedge B)$

$$
\begin{aligned}
\text{SA}: \quad & D_1 = A \\
\text{Ex. 4, } B := A \wedge B: \quad & D_2 = A \to (A \wedge B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_2): \quad & D_3 = A \wedge B \to A \wedge (A \wedge B) \\
\text{(A5b)}, \quad & \\
C := A \wedge (A \wedge B): \quad & D_4 = (A \wedge B \to A \wedge (A \wedge B)) \to (A \to (B \to A \wedge (A \wedge B))) \\
\text{MP}(D_3, D_4): \quad & D_5 = A \to (B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_5): \quad & D_6 = B \to A \wedge (A \wedge B)
\end{aligned}
$$

Here $\nvdash A \to (B \to A \wedge (A \wedge B))$
but $\vdash A \wedge A \to (B \to A \wedge (A \wedge B))$
$\vdash A \to (A \to (B \to A \wedge (A \wedge B)))$
$\vdash A \to (A \wedge B \to A \wedge (A \wedge B))$
(by substitution $B := A \wedge B$ in Example 4)

Exercise $\quad$ Prove in BL: $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

Example of deduction $\quad A \vdash B \to A \wedge (A \wedge B)$

$$
\begin{aligned}
\text{SA}: \quad & D_1 = A \\
\text{Ex. 4, } B := A \wedge B: \quad & D_2 = A \to (A \wedge B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_2): \quad & D_3 = A \wedge B \to A \wedge (A \wedge B) \\
\text{(A5b)}, \quad & \\
C := A \wedge (A \wedge B): \quad & D_4 = (A \wedge B \to A \wedge (A \wedge B)) \to (A \to (B \to A \wedge (A \wedge B))) \\
\text{MP}(D_3, D_4): \quad & D_5 = A \to (B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_5): \quad & D_6 = B \to A \wedge (A \wedge B)
\end{aligned}
$$

Here $\nvdash A \to (B \to A \wedge (A \wedge B))$
but $\vdash A \wedge A \to (B \to A \wedge (A \wedge B))$
$\vdash A \to (A \to (B \to A \wedge (A \wedge B)))$
$\vdash A \to (A \wedge B \to A \wedge (A \wedge B))$
(by substitution $B := A \wedge B$ in Example 4)

Exercise $\quad$ Prove in BL: $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$