Fuzzy intersection

is an operation on fuzzy sets defined using a fuzzy conjunction:

$$\mu_{A \underset{\cdot}{\cap} B}(x) = \mu_A(x) \underset{\cdot}{\wedge} \mu_B(x)$$

(we distinguish them by the same indices as the respective fuzzy conjunctions)

Theorem: The **standard** intersection is cut-consistent.

Proof: 1. Cutworthiness:

$$\begin{aligned}
\mathcal{R}_{A \underset{\mathrm{S}}{\cap} B}(\alpha) &= \{x \in X : \mu_{A \underset{\mathrm{S}}{\cap} B}(x) \geq \alpha\} \\
&= \{x \in X : (\mu_A(x) \geq \alpha) \wedge (\mu_B(x) \geq \alpha)\} \\
&= \{x \in X : \mu_A(x) \geq \alpha\} \cap \{x \in X : \mu_B(x) \geq \alpha\} \\
&= \mathcal{R}_A(\alpha) \cap \mathcal{R}_B(\alpha)
\end{aligned}$$

2. Cuts $\mathcal{R}_A(\alpha) \cap \mathcal{R}_B(\alpha)$ (for all $\alpha \in (0,1]$) determine a unique fuzzy set equal to $A \underset{\mathrm{S}}{\cap} B$.

Fuzzy disjunction (triangular conorm, t-conorm)

is a binary operation $\dot\vee : [0,1]^2 \to [0,1]$ such that

$$\alpha \mathbin{\dot\vee} \beta = \beta \mathbin{\dot\vee} \alpha \qquad \text{(commutativity)} \qquad \text{(S1)}$$

$$\alpha \mathbin{\dot\vee} (\beta \mathbin{\dot\vee} \gamma) = (\alpha \mathbin{\dot\vee} \beta) \mathbin{\dot\vee} \gamma \qquad \text{(associativity)} \qquad \text{(S2)}$$

$$\beta \le \gamma \Rightarrow \alpha \mathbin{\dot\vee} \beta \le \alpha \mathbin{\dot\vee} \gamma \qquad \text{(monotonicity)} \qquad \text{(S3)}$$

$$\alpha \mathbin{\dot\vee} 0 = \alpha \qquad \text{(boundary condition)} \qquad \text{(S4)}$$

Theorem: $\alpha \mathbin{\dot\vee} 1 = 1.$

Proof: $\alpha \mathbin{\dot\vee} 1 \overset{\text{(S3)}}{\ge} 0 \mathbin{\dot\vee} 1 \overset{\text{(S4)}}{=} 1.$

Examples of fuzzy disjunctions

- **Standard** (**max**, **Gödel**, **Zadeh** . . .):

$$\alpha \overset{\mathrm{S}}{\vee} \beta = \max(\alpha, \beta).$$

- **Łukasiewicz** (**Giles**, **bold**, **bounded sum** . . .):

$$\alpha \overset{\mathrm{L}}{\vee} \beta = \begin{cases} \alpha + \beta & \text{for } \alpha + \beta < 1, \\ 1 & \text{otherwise.} \end{cases}$$

- **Product** (**probabilistic** . . .):

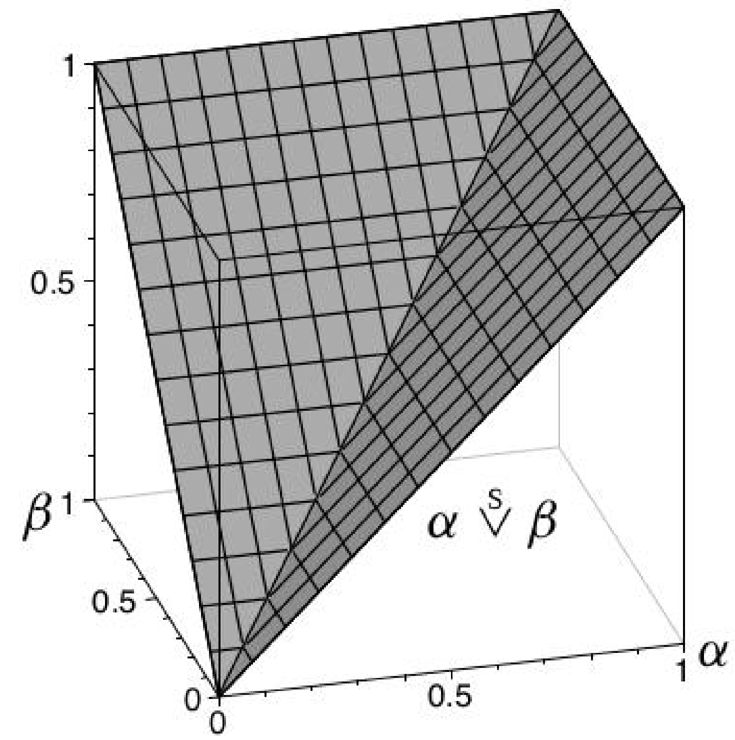
$$\alpha \overset{\mathrm{P}}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta.$$

- **Drastic** (**weak** . . .):

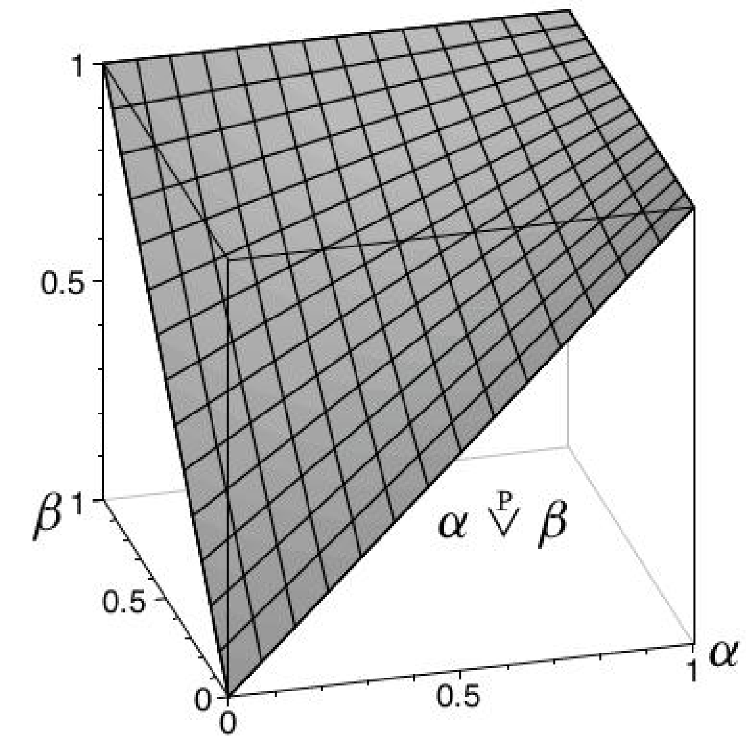
$$\alpha \overset{\mathrm{D}}{\vee} \beta = \begin{cases} \alpha & \text{for } \beta = 0, \\ \beta & \text{for } \alpha = 0, \\ 1 & \text{otherwise.} \end{cases}$$

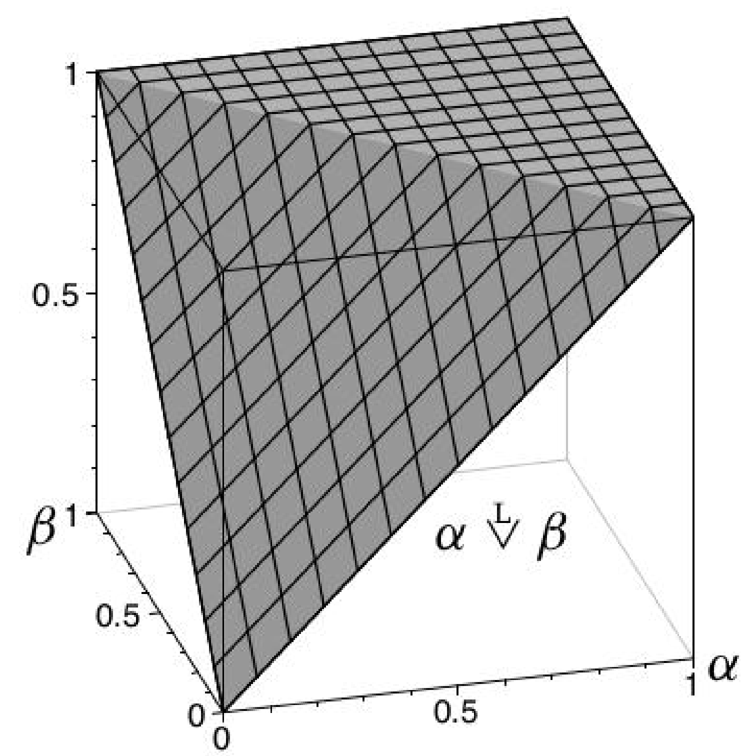
Basic fuzzy disjunctions

standard

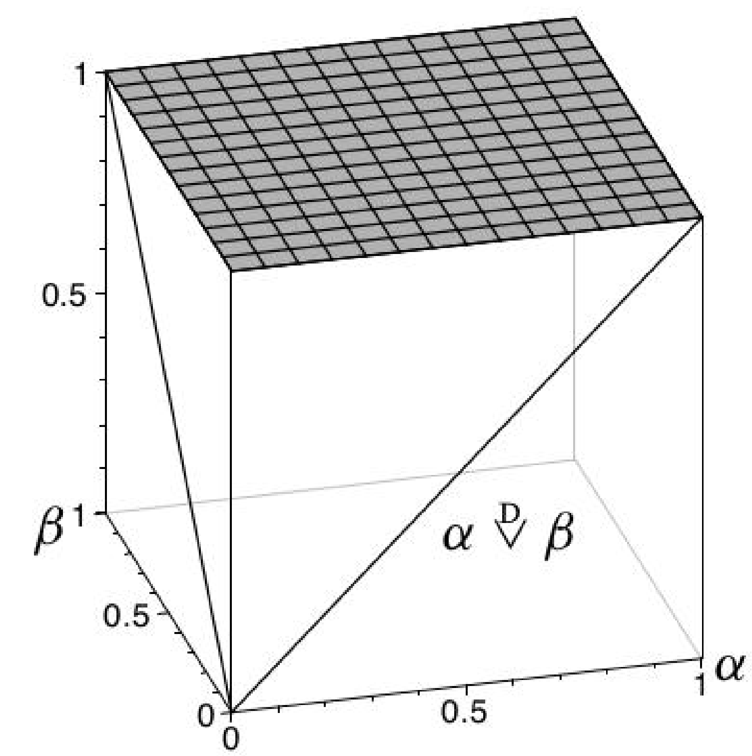


product





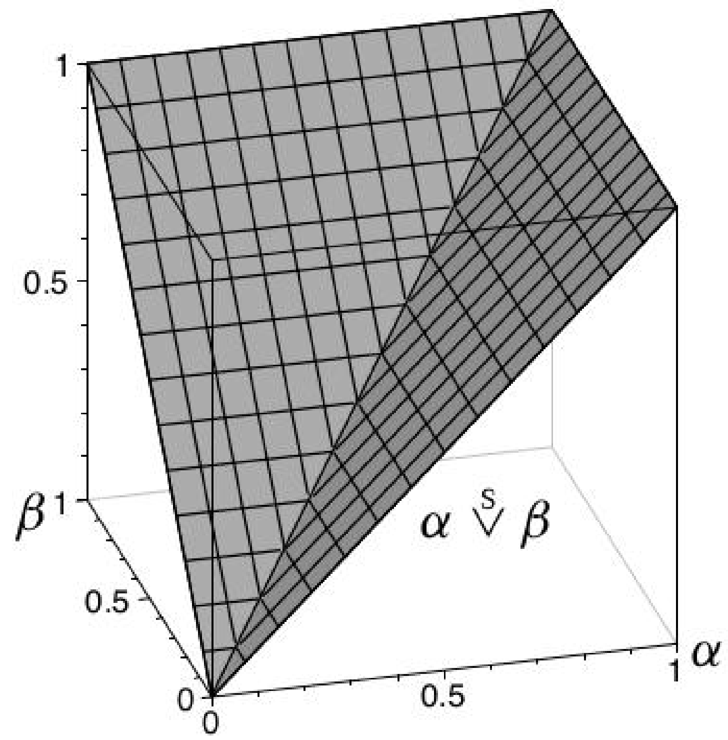
Łukasiewicz



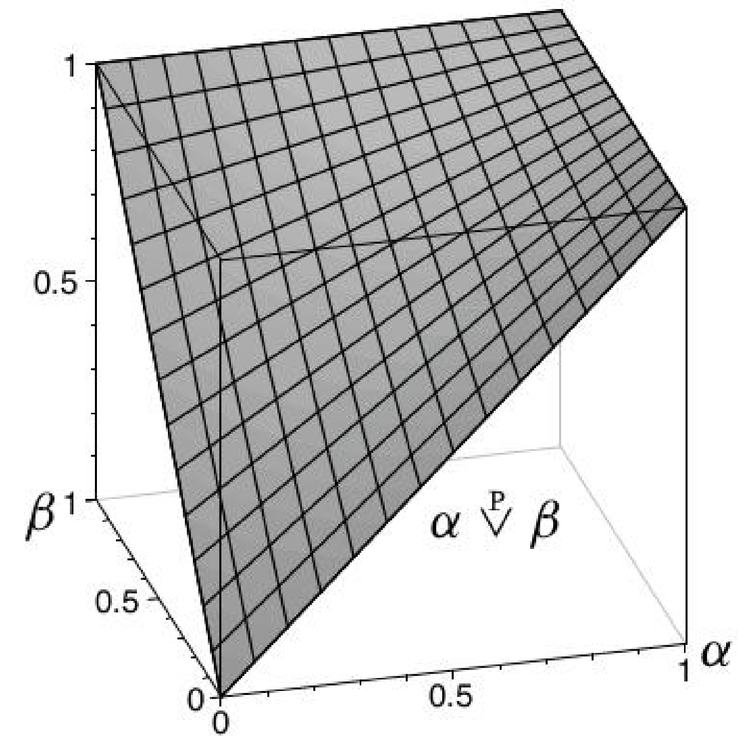
drastic

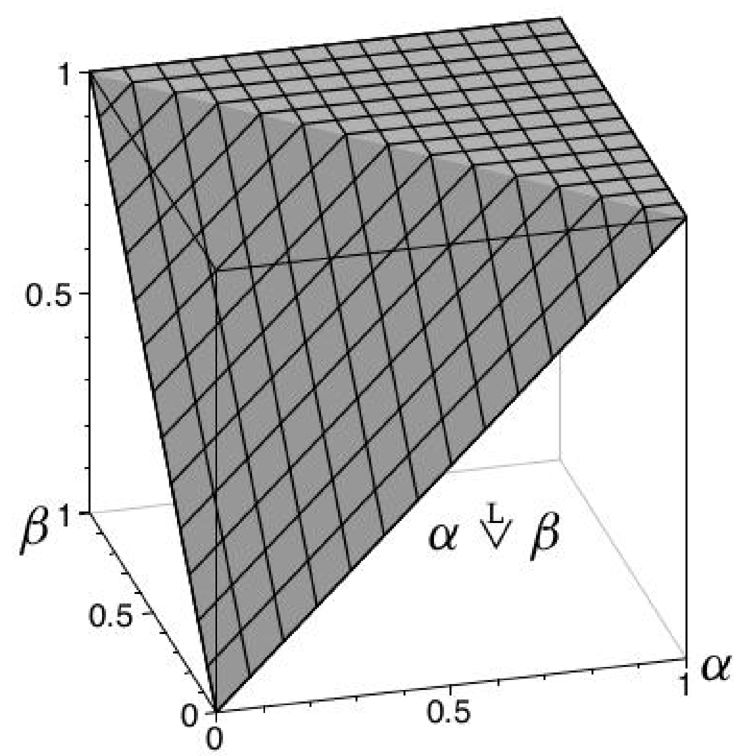
Basic fuzzy disjunctions

standard

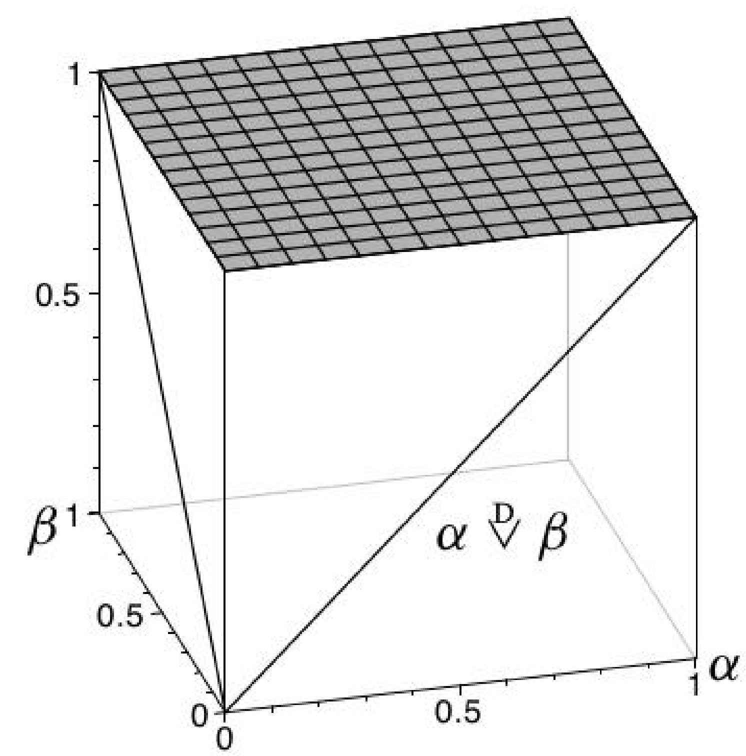


product





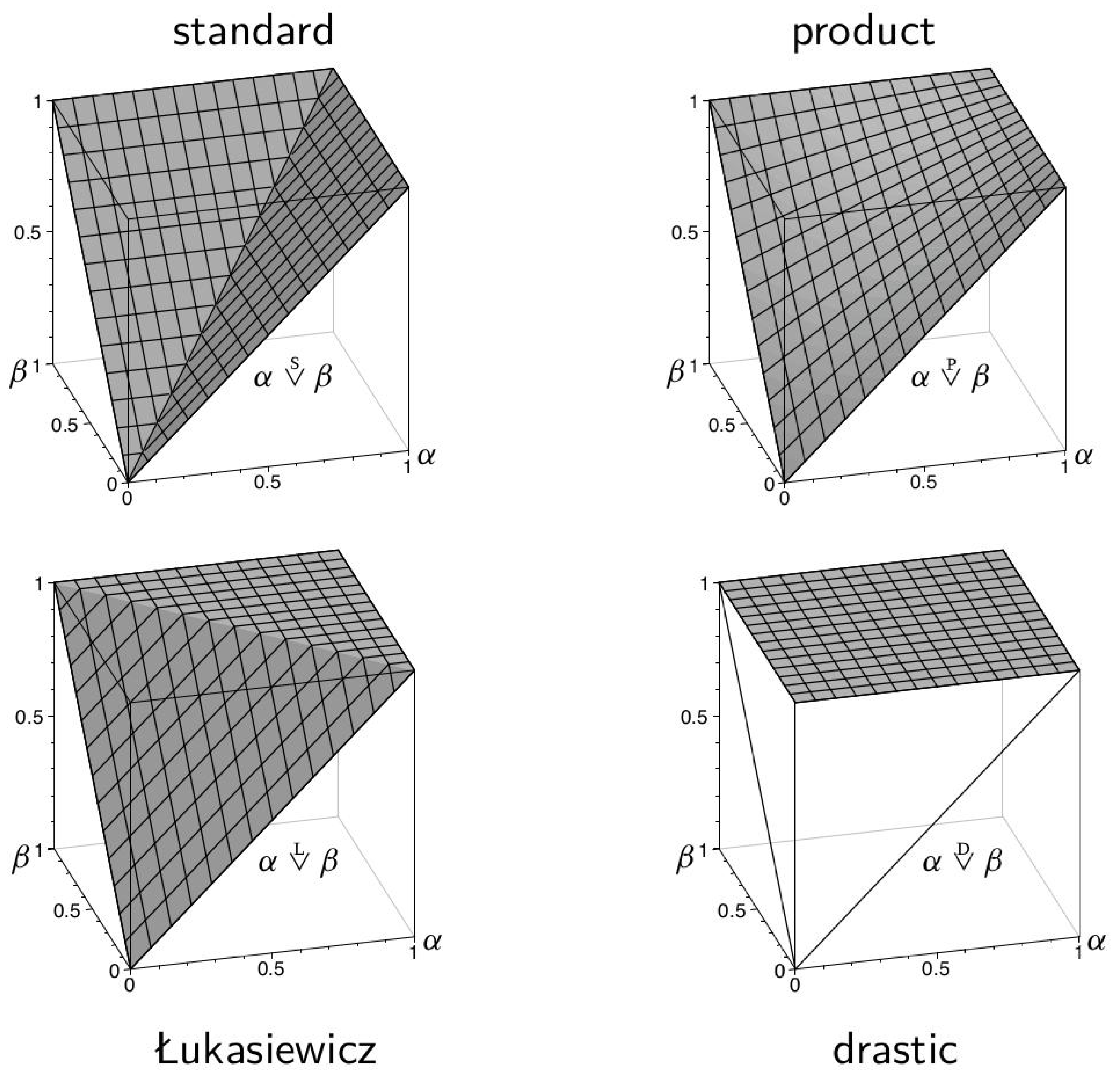
Łukasiewicz



drastic

- $$\alpha \overset{\mathrm{E}}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta}$$

Basic fuzzy disjunctions



- $$\alpha \overset{\mathrm{E}}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta}$$

Einstein fuzzy disjunction

- $$\alpha \overset{\mathrm{E}}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta}$$

Properties of fuzzy disjunctions

$$\forall \alpha, \beta \in [0,1]: \quad \alpha \overset{\mathrm{S}}{\vee} \beta \leq \alpha \dot{\vee} \beta \leq \alpha \overset{\mathrm{D}}{\vee} \beta.$$

The standard disjunction is the only one which is idempotent, i.e., $\alpha \dot{\vee} \alpha = \alpha$ for all $\alpha \in [0,1]$.

Duality

Let $\underset{\cdot}{\neg}$ be a fuzzy negation.

A. If $\underset{\cdot}{\wedge}$ is a fuzzy conjunction, then $\alpha \mathbin{\dot{\vee}} \beta = \underset{\cdot}{\neg}(\underset{\cdot}{\neg}\, \alpha \underset{\cdot}{\wedge} \underset{\cdot}{\neg}\, \beta)$ is a fuzzy disjunction (**dual** to $\underset{\cdot}{\wedge}$ with respect to $\underset{\cdot}{\neg}$).

B. If $\dot{\vee}$ is a fuzzy disjunction, then $\alpha \underset{\cdot}{\wedge} \beta = \underset{\cdot}{\neg}(\underset{\cdot}{\neg}\, \alpha \mathbin{\dot{\vee}} \underset{\cdot}{\neg}\, \beta)$ is a fuzzy conjunction (**dual** to $\dot{\vee}$ with respect to $\underset{\cdot}{\neg}$).

Theorem:

- The **Łukasiewicz** operations $\underset{\mathrm{L}}{\wedge}, \overset{\mathrm{L}}{\vee}$ are dual with respect to the **standard** negation.
- The **product** operations $\underset{\mathrm{P}}{\wedge}, \overset{\mathrm{P}}{\vee}$ are dual with respect to the **standard** negation.
- The **standard** operations $\underset{\mathrm{S}}{\wedge}, \overset{\mathrm{S}}{\vee}$ are dual with respect to **any** fuzzy negation.
- The **drastic** operations $\underset{\mathrm{D}}{\wedge}, \overset{\mathrm{D}}{\vee}$ are dual with respect to **any** fuzzy negation.

Classification of fuzzy disjunctions

A **continuous** fuzzy disjunction $\dot{\vee}$ is

- **Archimedean** if

$$\forall\alpha \in (0,1): \quad \alpha \mathbin{\dot{\vee}} \alpha > \alpha \tag{SA}$$

- **strict** if

$$\forall\alpha \in [0,1)\ \forall\beta,\gamma \in [0,1]: \quad \beta < \gamma \Rightarrow \alpha \mathbin{\dot{\vee}} \beta < \alpha \mathbin{\dot{\vee}} \gamma \tag{S3+}$$

- **nilpotent** if it is Archimedean and not strict.

Representation theorems for fuzzy disjunctions

Theorem: An operation $\dot{\vee} : [0,1]^2 \to [0,1]$ is a **strict** fuzzy disjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ such that

$$\alpha \mathbin{\dot{\vee}} \beta = i^{-1}\big(i(\alpha) \overset{\mathrm{P}}{\vee} i(\beta)\big).$$

Theorem: An operation $\dot{\vee} : [0,1]^2 \to [0,1]$ is a **nilpotent** fuzzy disjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**additive generator**) such that

$$\alpha \mathbin{\dot{\vee}} \beta = i^{-1}\big(i(\alpha) \overset{\mathrm{L}}{\vee} i(\beta)\big) = \begin{cases} i^{-1}\big(i(\alpha) + i(\beta)\big) & \text{if } i(\alpha) + i(\beta) \le 1 \\ 1 & \text{otherwise.} \end{cases}$$

Fuzzy union

is an operation on fuzzy sets defined using a fuzzy disjunction:

$$\mu_{A \dot{\cup} B}(x) = \mu_A(x) \dot{\vee} \mu_B(x).$$

(we distinguish them by the same indices as the respective fuzzy disjunctions)

Theorem: The **standard** union is cut-consistent.

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Fuzzy propositional algebras

equations written in **black** always hold
equations written in **red** hold for the standard fuzzy operations, but not for some others
equations written in **blue** hold only for some choices of fuzzy operations (not for the standard ones)

$$
\begin{array}{rclrcl}
\underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\alpha & = & \alpha, & & & \\
\alpha \dot{\vee} \beta & = & \beta \dot{\vee} \alpha, & \alpha \underset{\cdot}{\wedge} \beta & = & \beta \underset{\cdot}{\wedge} \alpha, \\
(\alpha \dot{\vee} \beta) \dot{\vee} \gamma & = & \alpha \dot{\vee} (\beta \dot{\vee} \gamma), & (\alpha \underset{\cdot}{\wedge} \beta) \underset{\cdot}{\wedge} \gamma & = & \alpha \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\wedge} \gamma), \\
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\alpha \dot{\vee} 1 & = & 1, & \alpha \underset{\cdot}{\wedge} 0 & = & 0, \\
\alpha \dot{\vee} 0 & = & \alpha, & \alpha \underset{\cdot}{\wedge} 1 & = & \alpha, \\
\alpha \underset{\cdot}{\wedge} \underset{\cdot}{\neg}\,\alpha & = & \mathbf{0}, & \alpha \dot{\vee} \underset{\cdot}{\neg}\,\alpha & = & \mathbf{1}, \\
\underset{\cdot}{\neg}(\alpha \underset{\cdot}{\wedge} \beta) & = & \underset{\cdot}{\neg}\,\alpha \dot{\vee} \underset{\cdot}{\neg}\,\beta, & \underset{\cdot}{\neg}(\alpha \dot{\vee} \beta) & = & \underset{\cdot}{\neg}\,\alpha \underset{\cdot}{\wedge} \underset{\cdot}{\neg}\,\beta.
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$$\begin{aligned}
\underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\alpha &= \alpha, & & \\
\alpha \dot\vee \beta &= \beta \dot\vee \alpha, & \alpha \underset{\cdot}{\wedge} \beta &= \beta \underset{\cdot}{\wedge} \alpha, \\
(\alpha \dot\vee \beta) \dot\vee \gamma &= \alpha \dot\vee (\beta \dot\vee \gamma), & (\alpha \underset{\cdot}{\wedge} \beta) \underset{\cdot}{\wedge} \gamma &= \alpha \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\wedge} \gamma), \\
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\alpha \dot\vee (\alpha \underset{\cdot}{\wedge} \beta) &= \alpha, & \alpha \underset{\cdot}{\wedge} (\alpha \dot\vee \beta) &= \alpha, \\
\alpha \dot\vee 1 &= 1, & \alpha \underset{\cdot}{\wedge} 0 &= 0, \\
\alpha \dot\vee 0 &= \alpha, & \alpha \underset{\cdot}{\wedge} 1 &= \alpha, \\
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\end{aligned}$$

Fuzzy implication

is any operation $\dot{\underset{\cdot}{\to}} : [0,1]^2 \to [0,1]$ which coincides with the classical implication on $\{0,1\}^2$.
We would like to satisfy the following properties, but we do not require them as axioms:

$$\alpha \dot{\underset{\cdot}{\to}} \beta = 1 \Leftarrow \alpha \le \beta, \tag{I1a}$$

$$\alpha \dot{\underset{\cdot}{\to}} \beta = 1 \Rightarrow \alpha \le \beta, \tag{I1b}$$

$$1 \dot{\underset{\cdot}{\to}} \beta = \beta, \tag{I2}$$

$\dot{\underset{\cdot}{\to}}$ is nonincreasing in the first argument and nondecreasing in the second, (I3)

$$\alpha \dot{\underset{\cdot}{\to}} \beta = \underset{S}{\neg}\, \beta \dot{\underset{\cdot}{\to}} \underset{S}{\neg}\, \alpha, \tag{I4}$$

$$\alpha \dot{\underset{\cdot}{\to}} (\beta \dot{\underset{\cdot}{\to}} \gamma) = \beta \dot{\underset{\cdot}{\to}} (\alpha \dot{\underset{\cdot}{\to}} \gamma), \tag{I5}$$

continuity. (I6)

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