Fuzzy conjunction (triangular norm, t-norm)

binary operation $\underset{\cdot}{\wedge}\colon [0,1]^2 \to [0,1]$ such that, for all $\alpha, \beta, \gamma \in [0,1]$:

$\alpha \underset{\cdot}{\wedge} \beta = \beta \underset{\cdot}{\wedge} \alpha$	(commutativity)	(T1)
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$\beta \leq \gamma \Rightarrow \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} \gamma$	(monotonicity)	(T3)
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Proposition: $\alpha \underset{\cdot}{\wedge} 0 = 0$.

Proof: Using (T3) and (T4): $\alpha \underset{\cdot}{\wedge} 0 \overset{\text{(T3)}}{\leq} 1 \underset{\cdot}{\wedge} 0 \overset{\text{(T4)}}{=} 0.$

Examples of fuzzy conjunctions

- **Standard** conjunction (**min**, **Gödel**, **Zadeh**, . . .):

$$\alpha \underset{\mathrm{S}}{\wedge} \beta = \min(\alpha, \beta).$$

- **Product** conjunction (**probabilistic**, **Goguen**, **algebraic product**, . . .):

$$\alpha \underset{\mathrm{P}}{\wedge} \beta = \alpha \cdot \beta.$$

- **Łukasiewicz** conjunction (**Giles**, **bold**, . . .):

$$\alpha \underset{\mathrm{L}}{\wedge} \beta = \begin{cases} \alpha + \beta - 1 & \text{if } \alpha + \beta - 1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- **Drastic** conjunction (**weak**, . . .):

$$\alpha \underset{\mathrm{D}}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1, \\ \beta & \text{if } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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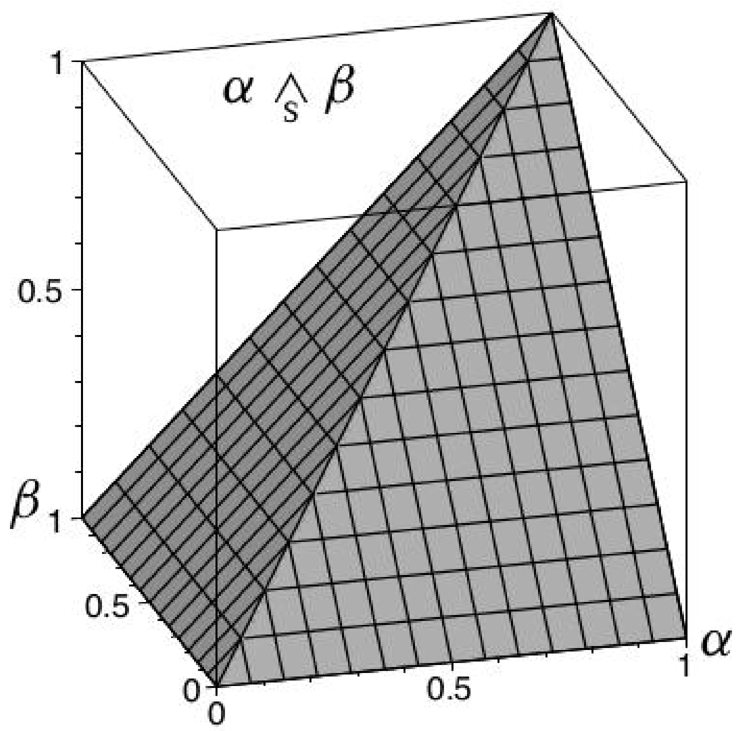
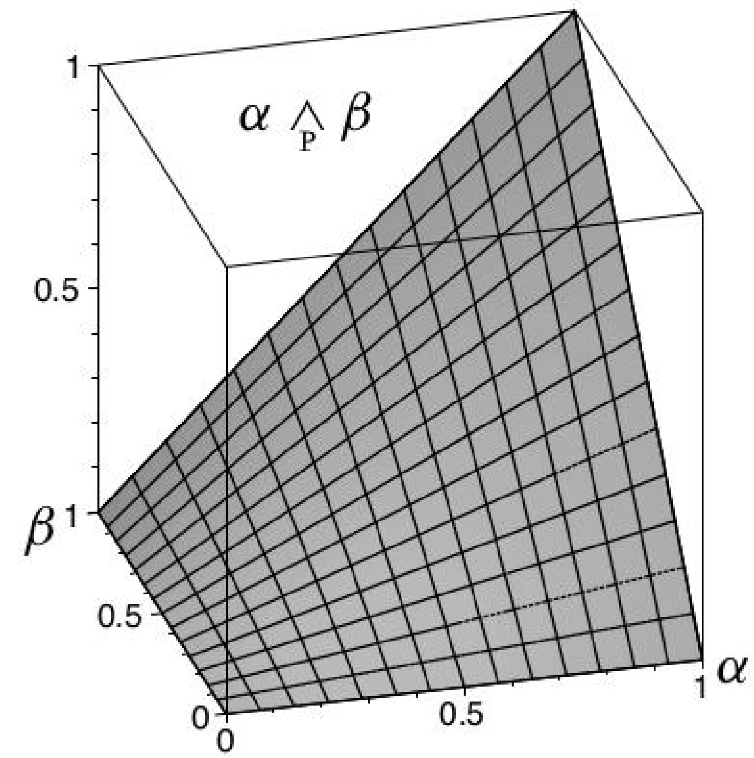
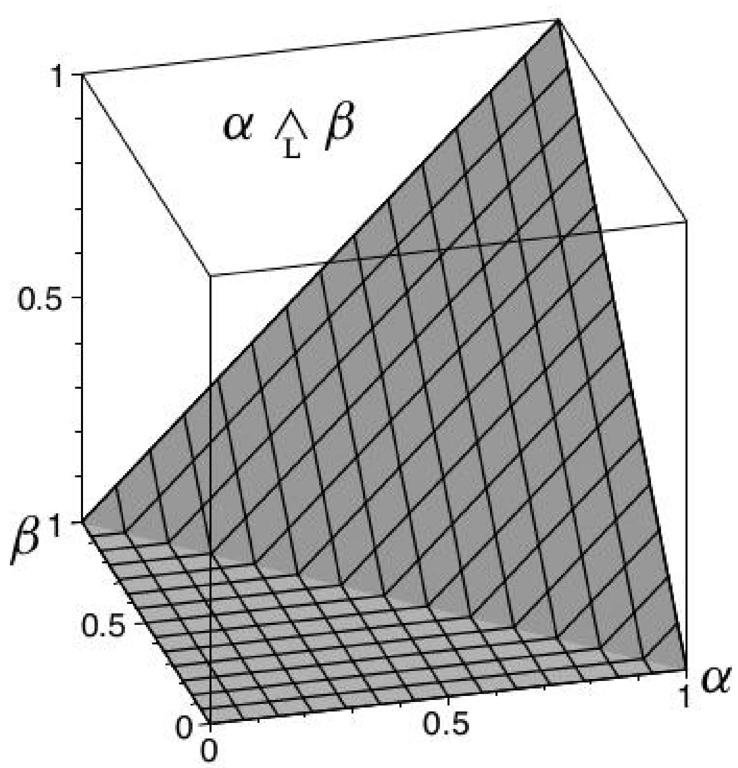
Basic fuzzy conjunctions

standard

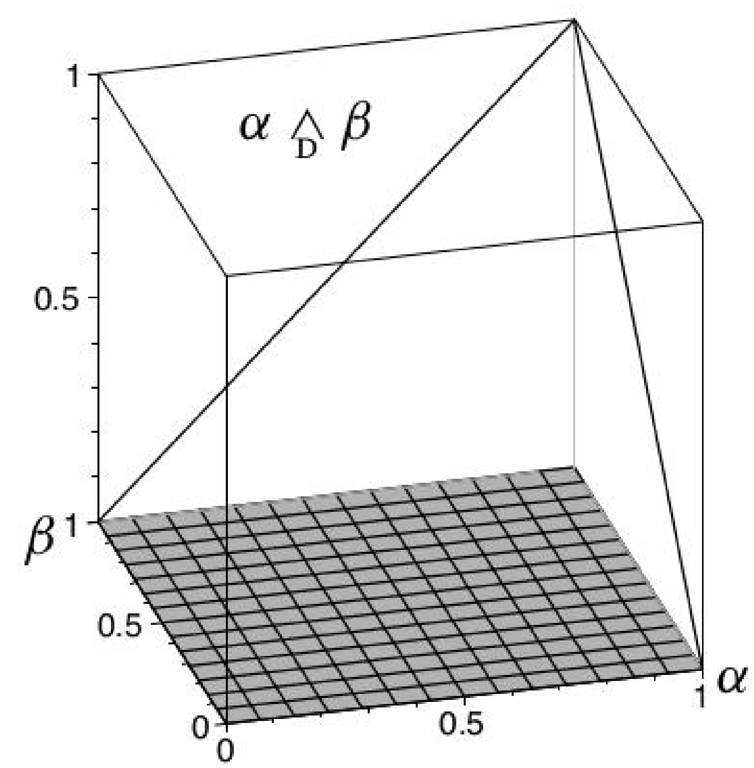


product





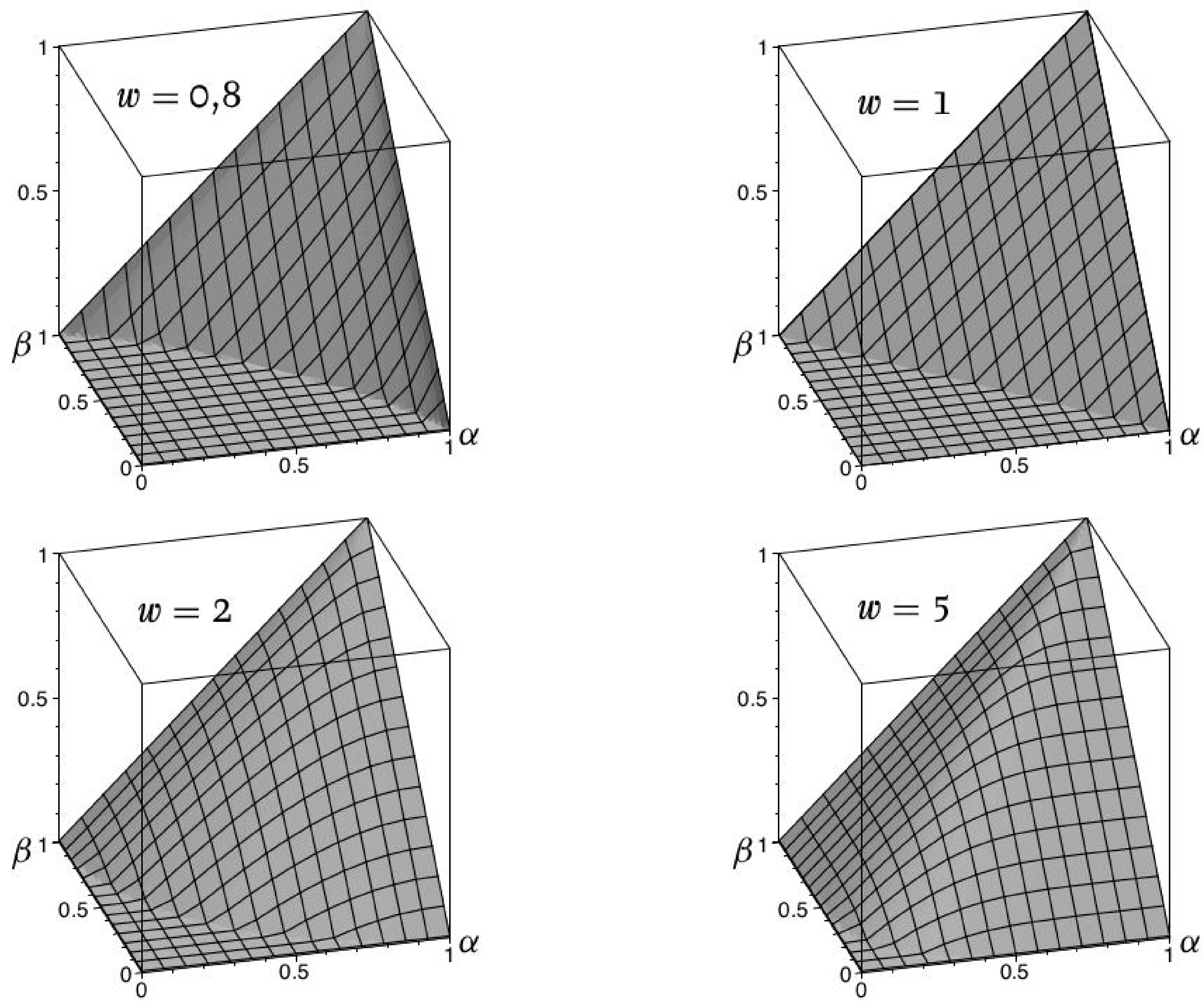
Łukasiewicz



drastic

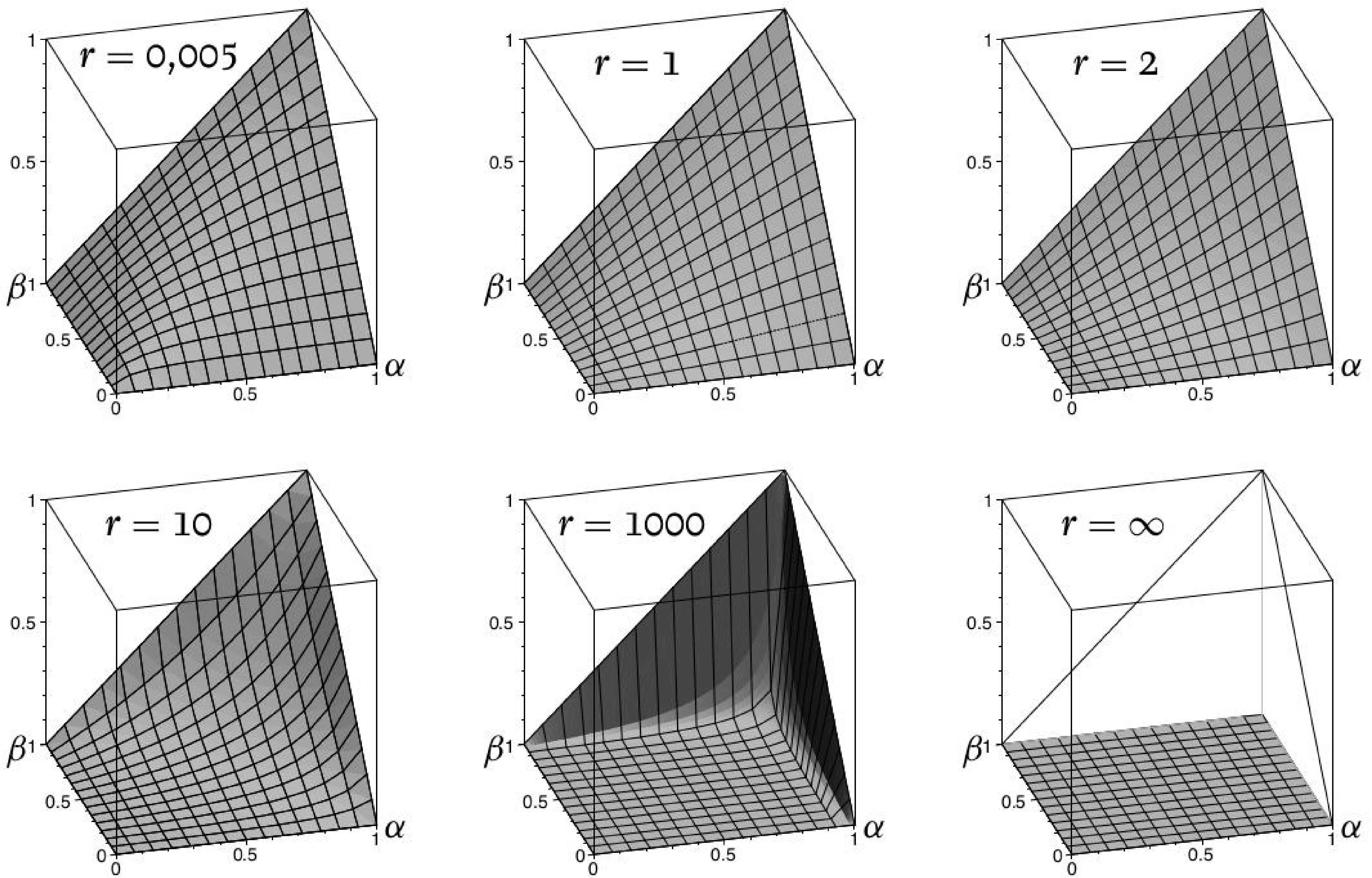
Yager fuzzy conjunctions

$$\alpha \underset{\mathrm{Y_w}}{\wedge} \beta = \max\left(1 - \left((\alpha - 1)^w + (\beta - 1)^w\right)^{\frac{1}{w}}, 0\right)$$



Hamacher fuzzy conjunctions

$$\alpha \bigwedge_{\mathrm{H_r}} \beta = \frac{\alpha\beta}{r + (1-r)(\alpha + \beta - \alpha\beta)}$$



Properties of fuzzy conjunctions

Proposition:

$$\forall \alpha, \beta \in [0,1] : \quad \alpha \underset{\mathrm{D}}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{\mathrm{S}}{\wedge} \beta.$$

Proof: If $\alpha = 1$ or $\beta = 1$, then (T4) gives the same result for all fuzzy conjunctions. Assume (without loss of generality) that $\alpha \leq \beta < 1$. Then

$$\alpha \underset{\mathrm{D}}{\wedge} \beta = 0 \leq \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} 1 = \alpha = \alpha \underset{\mathrm{S}}{\wedge} \beta.$$

Properties of fuzzy conjunctions

Proposition: Standard conjunction is the only one which is **idempotent**, i.e.,
$\forall \alpha \in [0,1] : \alpha \underset{\cdot}{\wedge} \alpha = \alpha$

Proof: Assume $\alpha, \beta \in [0,1]$, $\alpha \leq \beta$.

$$\alpha = \alpha \underset{\cdot}{\wedge} \alpha \overset{(T3)}{\leq} \alpha \underset{\cdot}{\wedge} \beta \overset{(T3)}{\leq} \alpha \underset{\cdot}{\wedge} 1 \overset{(T4)}{=} \alpha,$$

thus $\alpha \underset{\cdot}{\wedge} \beta = \alpha = \alpha \underset{S}{\wedge} \beta$.
Analogously for $\alpha > \beta$.

Representation of fuzzy conjunctions (in general)

Theorem: Let $\underset{1}{\wedge}$ be a fuzzy conjunction and $i\colon [0,1] \to [0,1]$ be an increasing bijection. Then the operation $\underset{2}{\wedge}\colon [0,1]^2 \to [0,1]$ defined by

$$\alpha \underset{2}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{1}{\wedge} i(\beta)\big)$$

is a fuzzy conjunction. If $\underset{1}{\wedge}$ is continuous, so is $\underset{2}{\wedge}$.

Proof:

- Commutativity (analogously for associativity):

$$\alpha \underset{2}{\wedge} \beta = i^{-1}(i(\alpha) \underset{1}{\wedge} i(\beta)) = i^{-1}(i(\beta) \underset{1}{\wedge} i(\alpha)) = \beta \underset{2}{\wedge} \alpha$$

- Monotonicity: Assume $\beta \le \gamma$.

$$\begin{aligned} i(\beta) &\le i(\gamma), \\ i(\alpha) \underset{1}{\wedge} i(\beta) &\le i(\alpha) \underset{1}{\wedge} i(\gamma), \\ \alpha \underset{2}{\wedge} \beta = i^{-1}(i(\alpha) \underset{1}{\wedge} i(\beta)) &\le i^{-1}(i(\alpha) \underset{1}{\wedge} i(\gamma)) = \alpha \underset{2}{\wedge} \gamma. \end{aligned}$$

- Boundary condition:

$$\alpha \underset{2}{\wedge} 1 = i^{-1}(i(\alpha) \underset{1}{\wedge} i(1)) = i^{-1}(i(\alpha) \underset{1}{\wedge} 1) = i^{-1}(i(\alpha)) = \alpha.$$

- Continuity: It is a composition of continuous functions.

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Equivalence of fuzzy conjunctions

Definition: We define a binary relation $\approx$ on fuzzy conjunctions such that

$$\underset{1}{\wedge} \approx \underset{2}{\wedge} \iff \exists i\colon [0,1] \to [0,1]\ \forall \alpha, \beta \in [0,1] : \alpha \underset{2}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{1}{\wedge} i(\beta)\big)\,.$$

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Transitivity: For $\quad \alpha \underset{2}{\wedge} \beta = i_1^{-1}\big(i_1(\alpha) \underset{1}{\wedge} i_1(\beta)\big)\,, \quad \alpha \underset{3}{\wedge} \beta = i_2^{-1}\big(i_2(\alpha) \underset{2}{\wedge} i_2(\beta)\big)\,,$
take the composition $i_3 = i_1 \circ i_2$, $\quad \alpha \underset{3}{\wedge} \beta = i_3^{-1}\big(i_3(\alpha) \underset{1}{\wedge} i_3(\beta)\big)\,.$

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We denote by $[\underset{\cdot}{\wedge}]$ the class of equivalence $\approx$ containing $\underset{\cdot}{\wedge}$.

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Proposition: $[\underset{\mathrm{S}}{\wedge}] = \{\underset{\mathrm{S}}{\wedge}\}$.

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Proposition: $[\underset{\mathrm{D}}{\wedge}] = \{\underset{\mathrm{D}}{\wedge}\}$.

Proposition: The set of all continuous fuzzy conjunctions is closed under $\approx$.

Classification of fuzzy conjunctions

Continuous fuzzy conjunction $\underset{\cdot}{\wedge}$ is

- **Archimedean** if
$$\forall \alpha \in (0,1): \quad \alpha \underset{\cdot}{\wedge} \alpha < \alpha \tag{TA}$$

- **strict** if
$$\forall \alpha \in (0,1]\ \forall \beta, \gamma \in [0,1]: \quad \beta < \gamma \Rightarrow \alpha \underset{\cdot}{\wedge} \beta < \alpha \underset{\cdot}{\wedge} \gamma \tag{T3+}$$

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Example: Product conjunction is strict, Łukasiewicz conjunction is nilpotent, standard and drastic conjunctions are not Archimedean (the standard one violates (TA), the drastic one is not continuous).

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$$\alpha \underset{1}{\wedge} \alpha < \alpha\,, \qquad \mid \alpha := i(\gamma)$$

$$i(\gamma) \underset{1}{\wedge} i(\gamma) < i(\gamma)\,, \qquad \mid i^{-1}$$

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Proposition: The class $[\underset{\mathrm{P}}{\wedge}]$ contains only strict conjunctions.

Equivalence of fuzzy conjunctions (continued)

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Proof:

$$
\begin{aligned}
0 &< \alpha\,, & \beta &< \gamma\,, \\
0 &< i(\alpha)\,, & i(\beta) &< i(\gamma)\,, \\
& & i(\alpha) \underset{\mathrm{P}}{\wedge} i(\beta) &< i(\alpha) \underset{\mathrm{P}}{\wedge} i(\gamma)\,, \\
& & \alpha \underset{\cdot}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{\mathrm{P}}{\wedge} i(\beta)\big) &< i^{-1}\big(i(\alpha) \underset{\mathrm{P}}{\wedge} i(\gamma)\big) = \alpha \underset{\cdot}{\wedge} \gamma\,.
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$$

Representation theorem for strict fuzzy conjunctions

Operation $\underset{\cdot}{\wedge}\colon [0,1]^2 \to [0,1]$ is a strict fuzzy conjunction iff there is an increasing bijection $i\colon [0,1] \to [0,1]$ (**multiplicative generator**) such that

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The proof of necessity is much more advanced.

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