Basic notions of a fuzzy controller

Rule database:

if X is A_1 **then** Y is C_1 **and**

$\ldots$

if X is A_n **then** Y is C_n

where

$X \in \mathcal{F}(\mathcal{X})$ is a fuzzy input

$Y = \Phi(X) \in \mathcal{F}(\mathcal{Y})$ is the corresponding fuzzy output

$A_i \in \mathcal{F}(\mathcal{X})$, $i = 1, \ldots, n$, are **antecedents** (**premises**) which can be interpreted as

- assumptions,
- domains of applicability, or
- typical fuzzy inputs

$C_i \in \mathcal{F}(\mathcal{Y})$, $i = 1, \ldots, n$, are **consequents** (**conclusions**) expressing the desired outputs

Dimensionality

Antecedents are subsets of multi-dimensional spaces
They carry information about several variables (and so do the consequents)
Usually they are decomposed to conjunctions (cylindric extensions) of one-dimensional fuzzy sets
Then the rules attain the form

if A_1 is A_{i1}
  and ...
  and A_μ is $A_{i\mu}$
then C_1 is C_{i1}
  and ...
  and C_ν is $C_{i\nu}$

$i = 1, \dots, n$

If an antecedent has a more complex shape (non-convex), we may cover it approximately by several rules of the above form

Simplifying assumptions

1. We ignore the conjunctions (cylindric extensions) and admit arbitrary shapes of antecedents

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1. We ignore the conjunctions (cylindric extensions) and admit arbitrary shapes of antecedents
2. We decompose the output to single variables considered independently. Without loss of generality, we restrict attention to MISO (Mulitple Input Single Output) systems

Compositional rule of inference

The rule base is represented by a fuzzy relation $R \in \mathcal{F}(\mathcal{X} \times \mathcal{Y})$
The output, Y, is obtained by a composition of R with the input, X:

$$Y = \Phi(X) = X \underset{\cdot}{\circ} R$$

$$Y(y) = \sup_{x \in \mathcal{X}} \big(R(x, y) \underset{\cdot}{\wedge} X(x)\big)$$

where $\underset{\cdot}{\wedge}$ is a t-norm (fuzzy conjunction); different choices are possible, but we shall restrict to **continuous t-norms**

The supremum is the standard t-conorm; it should not be replaced by another t-conorm (because it may have uncountably many arguments)

How to derive the fuzzy relation R from the rule base

The most natural idea: **Residuum-based fuzzy controller**:

$$R_{\mathsf{RES}}(x,y) = \min_i \big(A_i(x) \underset{\cdot}{\rightarrow} C_i(y)\big)$$

where $\underset{\cdot}{\rightarrow}$ is a **fuzzy implication**, usually the **residuum** (**R-implication**) of $\underset{\cdot}{\wedge}$,

$$\alpha \underset{\cdot}{\rightarrow} \beta = \sup\{\gamma \in [0,1] \mid \gamma \underset{\cdot}{\wedge} \alpha \le \beta\}$$

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Properties of residua of continuous t-norms:

- $\alpha \underset{\cdot}{\rightarrow} \beta = 1$ iff $\alpha \leq \beta$
- $1 \underset{\cdot}{\rightarrow} \beta = \beta$
- non-increasing in the first and non-decreasing in the second variable
- continuous iff the t-norm $\underset{\cdot}{\wedge}$ is nilpotent
- **adjointness**: $a \underset{\cdot}{\wedge} b \leq c$ iff $a \leq b \underset{\cdot}{\rightarrow} c$

How to derive the fuzzy relation R from the rule base 2

Mamdani–Assilian fuzzy controller:

$$R_{\mathsf{MA}}(x,y) = \max_i \big(A_i(x) \mathbin{\dot\wedge} C_i(y)\big)$$

Logically this is a **disjunction of conjunctions**, not a conjunction of implications

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These expressions are not totally different; if **crisp** sets A_i, $i = 1, \ldots, n$, form a partition of $\mathcal{X}$ (i.e., they are mutually disjoint and $\bigcup_i A_i = \mathcal{X}$), then $R_{\mathsf{MA}} = R_{\mathsf{RES}}$

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However, this usually is not the case

Comparison of residuum-based and Mamdani–Assilian controllers

Continuity:

R_{RES} only for $\underset{\cdot}{\wedge}$ nilpotent

R_{MA} always

Computational efficiency:

$$\Phi_{\mathsf{RES}}(X)(y) = \sup_x \Big(X(x) \underset{\cdot}{\wedge} \min_i (A_i(x) \underset{\cdot}{\to} C_i(y)) \Big)$$

requires **three** nested cycles (over $\mathcal{X}$ and $\mathcal{Y}$ and over the number of rules)

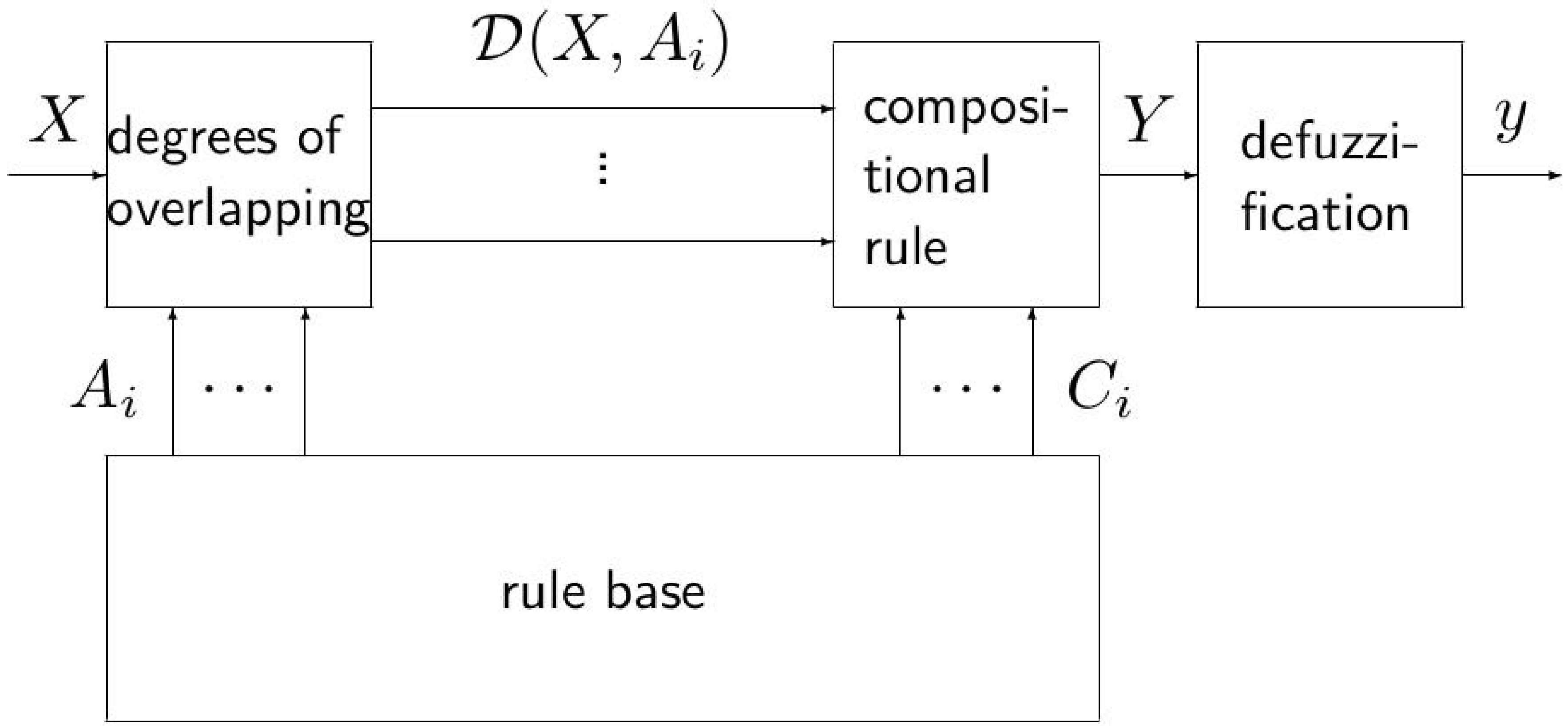
$$\begin{aligned}\Phi_{\mathsf{MA}}(X)(y) &= \sup_x \Big(X(x) \underset{\cdot}{\wedge} \max_i (A_i(x) \underset{\cdot}{\wedge} C_i(y)) \Big) \\ &= \max_i \sup_x \Big(X(x) \underset{\cdot}{\wedge} A_i(x) \underset{\cdot}{\wedge} C_i(y) \Big) \\ &= \max_i (\mathcal{D}(X, A_i) \underset{\cdot}{\wedge} C_i(y))\end{aligned}$$

$\mathcal{D}(X, A_i) = \sup_x (X(x) \underset{\cdot}{\wedge} A_i(x))$ ... the **degree of overlapping (non-disjointness)** here equal to the **degree of firing (applicability)**

requires **two** nested cycles (over $\mathcal{X}$ and the number of rules) resulting in real numbers $\mathcal{D}(X, A_i)$, $i = 1, \ldots, n$; **then two** nested cycles (over $\mathcal{Y}$ and the number of rules)

Φ_{MA} can be computed more efficiently (approx. $\#Y/2$-times faster)

Principle of Mamdani–Assilian controller — block diagram



Generalized Compositional Rule of Inference

[Thiele 1995], [Lehmke, Reusch, Temme, Thiele 1998]
FATI principle (First Aggregation, Then Inference)

$$R_{\mathsf{FATI}}(x,y) = \beta\big(\pi_1\big(A_1(x), C_1(y)\big), \ldots, \pi_n\big(A_n(x), C_n(y)\big)\big)$$

where $\pi_i\colon [0,1]^2 \to [0,1]$, $\beta\colon [0,1]^n \to [0,1]$

$$Y(y) = \Phi_{\mathsf{FATI}}(X)(y) = Q\{\kappa\big(X(x), R_{\mathsf{FATI}}(x,y)\big) \mid x \in \mathcal{X}\}$$

where $\kappa\colon [0,1]^2 \to [0,1]$, $Q\colon \mathcal{P}([0,1]) \to [0,1]$ (almost arbitrary operations)

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Mamdani–Assilian controller: $\pi_i = \underset{\cdot}{\wedge}, \quad \beta = \max, \quad \kappa = \underset{\cdot}{\wedge}, \quad Q = \sup$

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Requirements on the rule base [Driankov et al. 1993]

1. **Completeness**: $\bigcup_i \operatorname{Supp} A_i = \mathcal{X}$, where Supp $A_i = \{x \in \mathcal{X} \mid A_i(x) > 0\}$

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   The output of the controller should be the fuzzy union of the outputs of separate rules (i.e., FATI=FITA;
   this weaker form always holds for a Mamdani–Assilian controller;
   see also [Lehmke, Reusch, Temme, Thiele 1998] for more general sufficient conditions for this equality)

Recommendations on the rule base [Driankov et al. 1993]

Antecedents (one-dimensional) should be

- **normal**, $\forall i \; \exists x \in \mathcal{X} : A_i(x) = 1$
- continuous
- symmetric (when possible, usually not at the borders of the input space!)

The recommended degree of overlapping of neighbouring antecedents (computed using the standard t-norm, $\min$) is 0.5

The recommended endpoints of the support of an antecedent are the peeks of the neighbouring antecedents

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- **Crisp correctness (crisp interaction)**: $\big(A_i(x) = 1\big) \Rightarrow \big(\Phi(x) = \Phi(\{x\}) = C_i\big)$ ("if there is a totally firing rule, it determines the output")