Fuzzy control

Mirko Navara
http://cmp.felk.cvut.cz/~navara

Outline:

- Historical introduction
- Brief overview of classical control theory
- Principles of fuzzy control

Basics of classical control theory

Intuitively used in ancient times

Watt – steam engine

negative feedback:

high speed $\Rightarrow$ less steam

low speed $\Rightarrow$ more steam

Basics of classical control theory

Intuitively used in ancient times

Watt – steam engine

negative feedback:

high speed $\Rightarrow$ less steam

low speed $\Rightarrow$ more steam

Watt did not care of non-linearity and dynamical properties of the controller

Basics of classical control theory

Intuitively used in ancient times

Watt – steam engine

negative feedback:
high speed $\Rightarrow$ less steam
low speed $\Rightarrow$ more steam

Watt did not care of non-linearity and dynamical properties of the controller

It sufficed to have it very sensitive and much faster than the controlled system

Middle of 20th century: [Wiener, Shannon, Nyquist, ... , Zadeh]

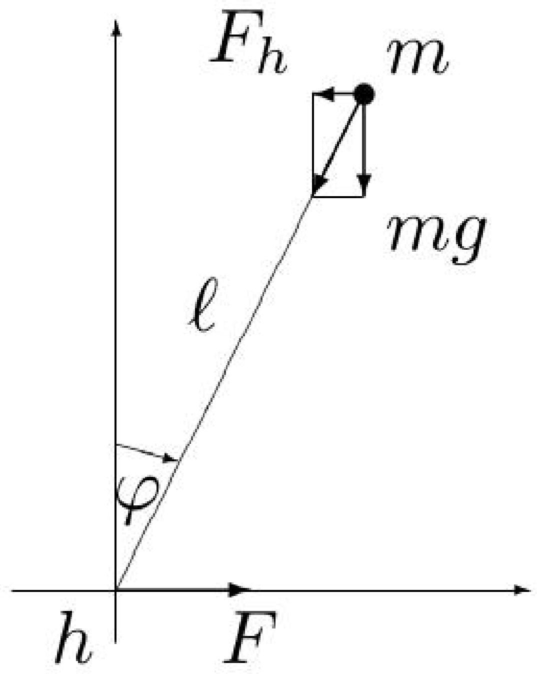
Inspiration: Cartpole problem (inverted pendulum)

Very simplified model:

- no friction ($\Rightarrow$ no damping)
- zero moment of inertia (**single pendulum**); a real pendulum is described by the same model with some **equivalent length**
- the influence of the mass of the pendulum on our movements is neglected
- **linearization** around $\varphi = 0$
- the acceleration (and the fluctuations of forces) in the vertical direction are neglected

---

Inspiration: Cartpole problem (inverted pendulum)



Constants:
ℓ = lenght of the pendulum
m = mass of the pendulum
g = acceleration of gravity

Variables:
t = time
$h(t)$ = horizontal coordinate of the axis of the pendulum
$\varphi(t)$ = angle of the pendulum (measured from the vertical direction)
$a(t)$ = acceleration of the axis of the pendulum in the horizontal direction
(proportional to the acting force $F(t)$)
$q(t) = h(t) + \ell \sin \varphi(t)$ = horizontal coordinate of the mass of the pendulum

Inspiration: Cartpole problem (inverted pendulum)

Equations (with (t) omitted):

$$q = h + \ell \sin\varphi$$

Linearized coordinate of the mass of the pendulum:

$$q = h + \ell\,\varphi$$

Its second derivative is proportional to the horizontal force on the pendulum:

$$\frac{F_h}{m} = \ddot{h} + \ell\,\ddot{\varphi}$$

The pendulum transfers only a force parallel to it, hence

$$\frac{F_h}{mg} = \tan\varphi \qquad \text{linearized:} \qquad \frac{F_h}{m} = g\varphi$$

We act through a force causing an acceleration

$$a = \ddot{h}$$

The dynamics of the system is described by a system of linear ODEs:

$$\begin{aligned} \ddot{h} + \ell\,\ddot{\varphi} &= g\,\varphi \\ \ddot{h} &= a \end{aligned}$$

Inspiration: Cartpole problem (inverted pendulum)

State variables:

$$x_1 = \varphi \qquad x_2 = \dot{\varphi} \qquad x_3 = h \qquad x_4 = \dot{h} \qquad \mathbf{x} = \begin{bmatrix} \varphi \\ \dot{\varphi} \\ h \\ \dot{h} \end{bmatrix}$$

Control variable (input):

$$u_1 = a \qquad \mathbf{u} = \begin{bmatrix} a \end{bmatrix}$$

The linearized dynamics of the system is described by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \dot{x}_1 = \dot{\varphi} &= x_2 \\ \dot{x}_2 = \ddot{\varphi} &= \frac{g}{\ell}\varphi - \frac{1}{\ell}a = \frac{g}{\ell}x_1 - \frac{1}{\ell}u_1 \\ \dot{x}_3 = \dot{h} &= x_4 \\ \dot{x}_4 = \ddot{h} &= a = u_1 \end{aligned}$$

Inspiration: Cartpole problem (inverted pendulum)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{-1}{\ell} \\ 0 \\ 1 \end{bmatrix}$$

Inspiration: Cartpole problem (inverted pendulum)

State variables:

$$x_1 = \varphi \qquad x_2 = \dot{\varphi} \qquad x_3 = h \qquad x_4 = \dot{h} \qquad \mathbf{x} = \begin{bmatrix} \varphi \\ \dot{\varphi} \\ h \\ \dot{h} \end{bmatrix}$$

Control variable (input):

$$u_1 = a \qquad \mathbf{u} = \begin{bmatrix} a \end{bmatrix}$$

The linearized dynamics of the system is described by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \dot{x}_1 = \dot{\varphi} &= x_2 \\ \dot{x}_2 = \ddot{\varphi} &= \frac{g}{\ell}\varphi - \frac{1}{\ell}a = \frac{g}{\ell}x_1 - \frac{1}{\ell}u_1 \\ \dot{x}_3 = \dot{h} &= x_4 \\ \dot{x}_4 = \ddot{h} &= a = u_1 \end{aligned}$$

Inspiration: Cartpole problem (inverted pendulum)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{-1}{\ell} \\ 0 \\ 1 \end{bmatrix}$$

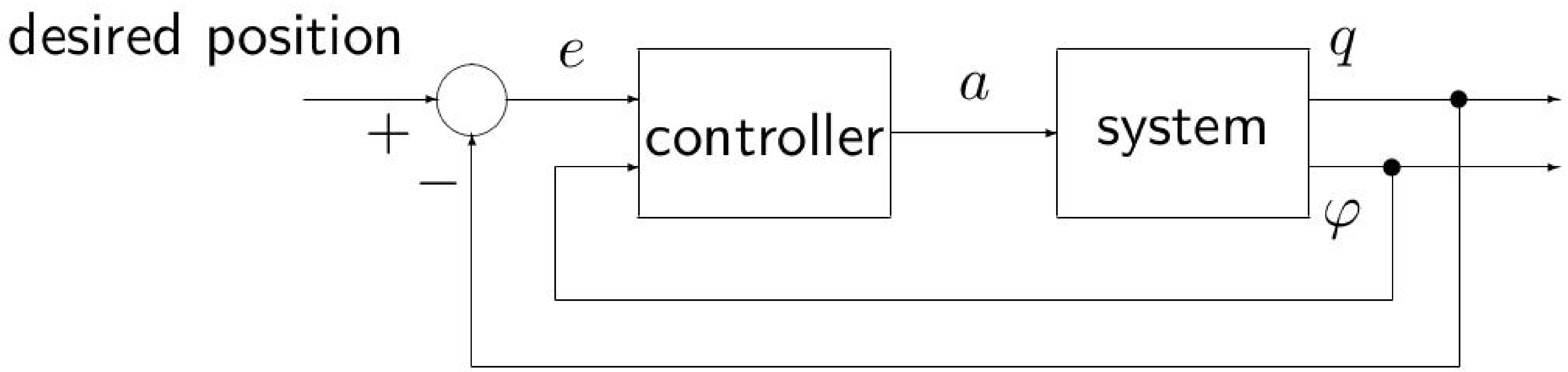
Inspiration: Cartpole problem (inverted pendulum)

Output variables:

$$y_1 = \varphi \qquad y_2 = q = h + \ell\varphi \qquad \mathbf{y} = \begin{bmatrix} \varphi \\ q \end{bmatrix} \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

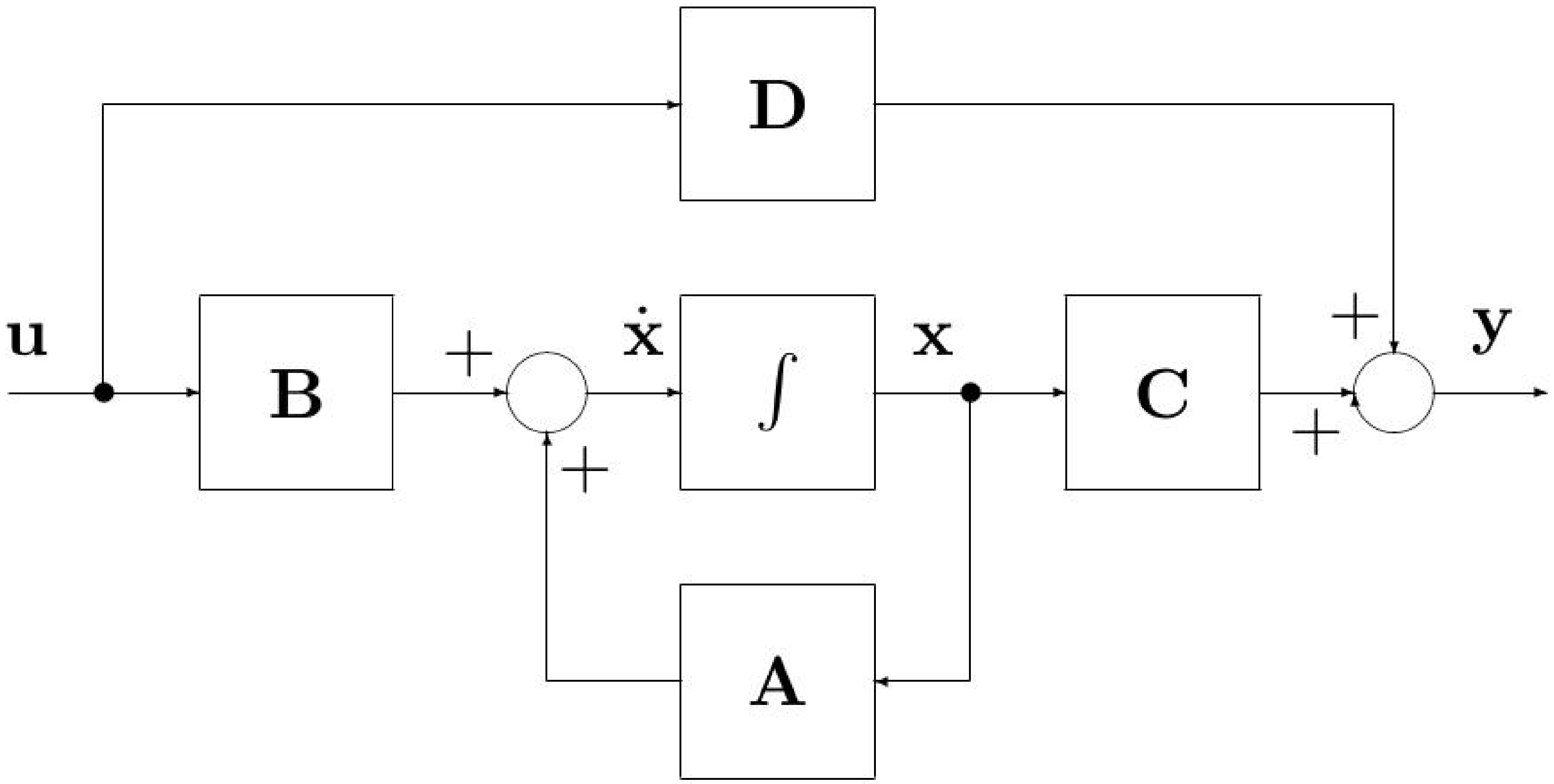
(no direct influence of the control on the output)



Descriptions of a linear dynamical system

Internal description: $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$



External description: using the Laplace images $\mathbf{U}(s), \mathbf{Y}(s)$ of $\mathbf{u}(t), \mathbf{y}(t)$ (with zero initial conditions)

$\mathbf{Y}(s) = \mathbf{G}(s)\,\mathbf{U}(s)$ where $\mathbf{G}(s)$ is the **transfer function** of the system

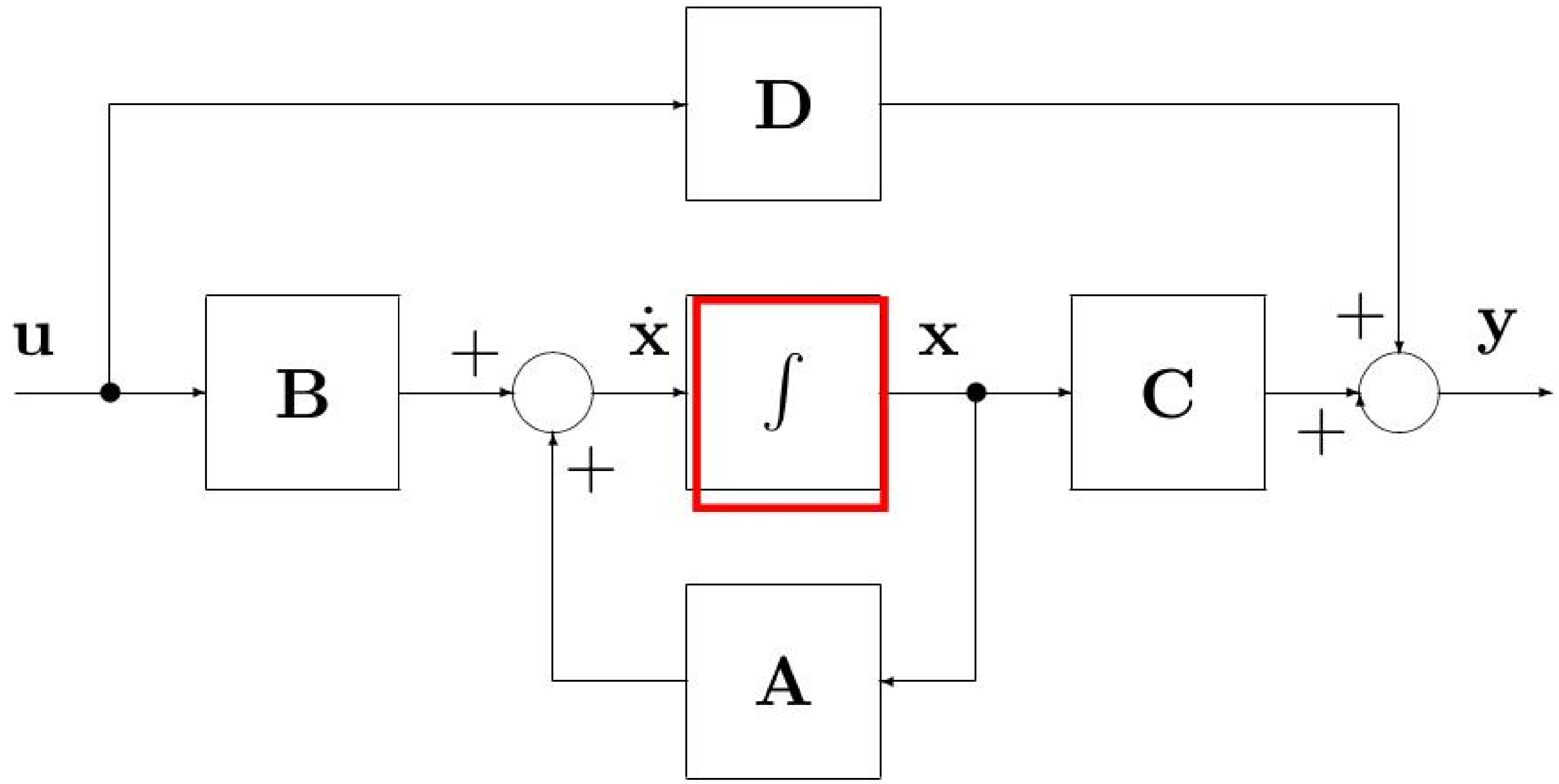
Having vectors of variables, $\mathbf{G}(s)$ is a matrix of functions of s

Its element $\mathbf{G}_{ij}(s)$ is the Laplace image of the response at the jth output to the Dirac pulse at the ith input (difficult to measure directly)

Descriptions of a linear dynamical system

Internal description: $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$



External description: using the Laplace images $\mathbf{U}(s), \mathbf{Y}(s)$ of $\mathbf{u}(t), \mathbf{y}(t)$ (with zero initial conditions)

$\mathbf{Y}(s) = \mathbf{G}(s)\,\mathbf{U}(s)$ where $\mathbf{G}(s)$ is the **transfer function** of the system

Having vectors of variables, $\mathbf{G}(s)$ is a matrix of functions of s

Its element $\mathbf{G}_{ij}(s)$ is the Laplace image of the response at the jth output to the Dirac pulse at the ith input (difficult to measure directly)

How to derive the external description from the internal one?

Under zero initial conditions ($\mathbf{I}$ denotes the unit matrix):

$$
\begin{aligned}
s\,\mathbf{X}(s) &= \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s) \\
\mathbf{Y}(s) &= \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{X}(s) &= (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s) \\
\mathbf{Y}(s) &= \underbrace{\left(\mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(s)}\,\mathbf{U}(s)
\end{aligned}
$$

$\det(s\,\mathbf{I} - \mathbf{A}) = 0$ is the **characteristic equation** of the system

Its solutions (in variable s) are **eigenvalues**

Each of them corresponds to one exponential component of the solution

The system is stable iff all characteristic numbers have negative real parts

(this can be tested without finding the characteristic numbers, which is a difficult problem)

How to derive the internal description from the external one?

This is not unique, there are many standardized methods whose choice depends on the hardware implementation

---

How to derive the external description from the internal one?

Under zero initial conditions ($\mathbf{I}$ denotes the unit matrix):

$$
\begin{aligned}
s\,\mathbf{X}(s) &= \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s) \\
\mathbf{Y}(s) &= \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s)
\end{aligned}
$$

sX(s) - AX(s) = BU(s)

$$
\begin{aligned}
\mathbf{X}(s) &= (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s) \\
\mathbf{Y}(s) &= \underbrace{\left(\mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(s)}\,\mathbf{U}(s)
\end{aligned}
$$

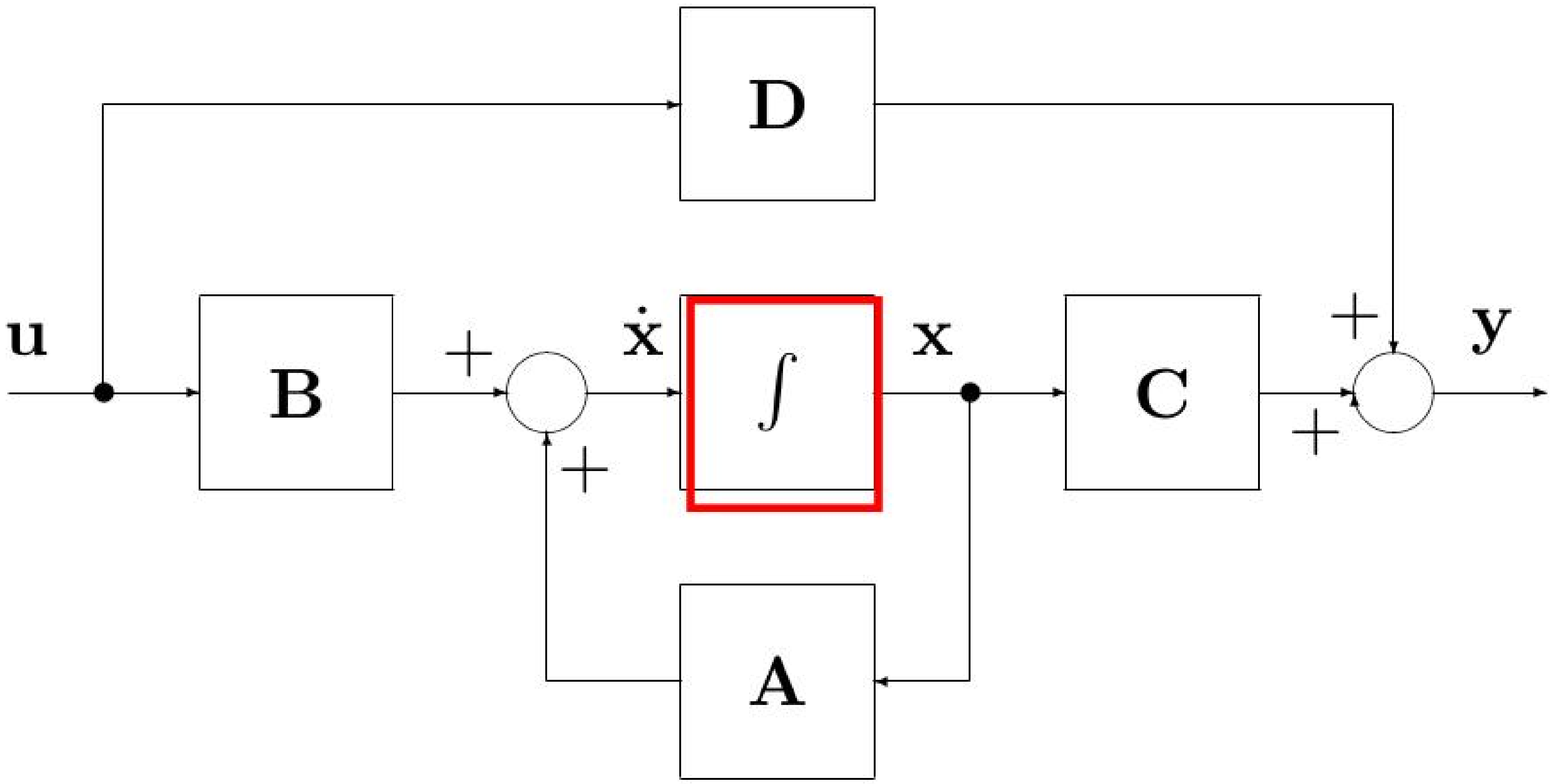
$\det(s\,\mathbf{I} - \mathbf{A}) = 0$ is the **characteristic equation** of the system
Its solutions (in variable s) are **eigenvalues**
Each of them corresponds to one exponential component of the solution
The system is stable iff all characteristic numbers have negative real parts
(this can be tested without finding the characteristic numbers, which is a difficult problem)

How to derive the internal description from the external one?
This is not unique, there are many standardized methods whose choice depends on the hardware implementation

Descriptions of a linear dynamical system

Internal description: $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$



External description: using the Laplace images $\mathbf{U}(s), \mathbf{Y}(s)$ of $\mathbf{u}(t), \mathbf{y}(t)$ (with zero initial conditions)

$\mathbf{Y}(s) = \mathbf{G}(s)\,\mathbf{U}(s)$ where $\mathbf{G}(s)$ is the **transfer function** of the system

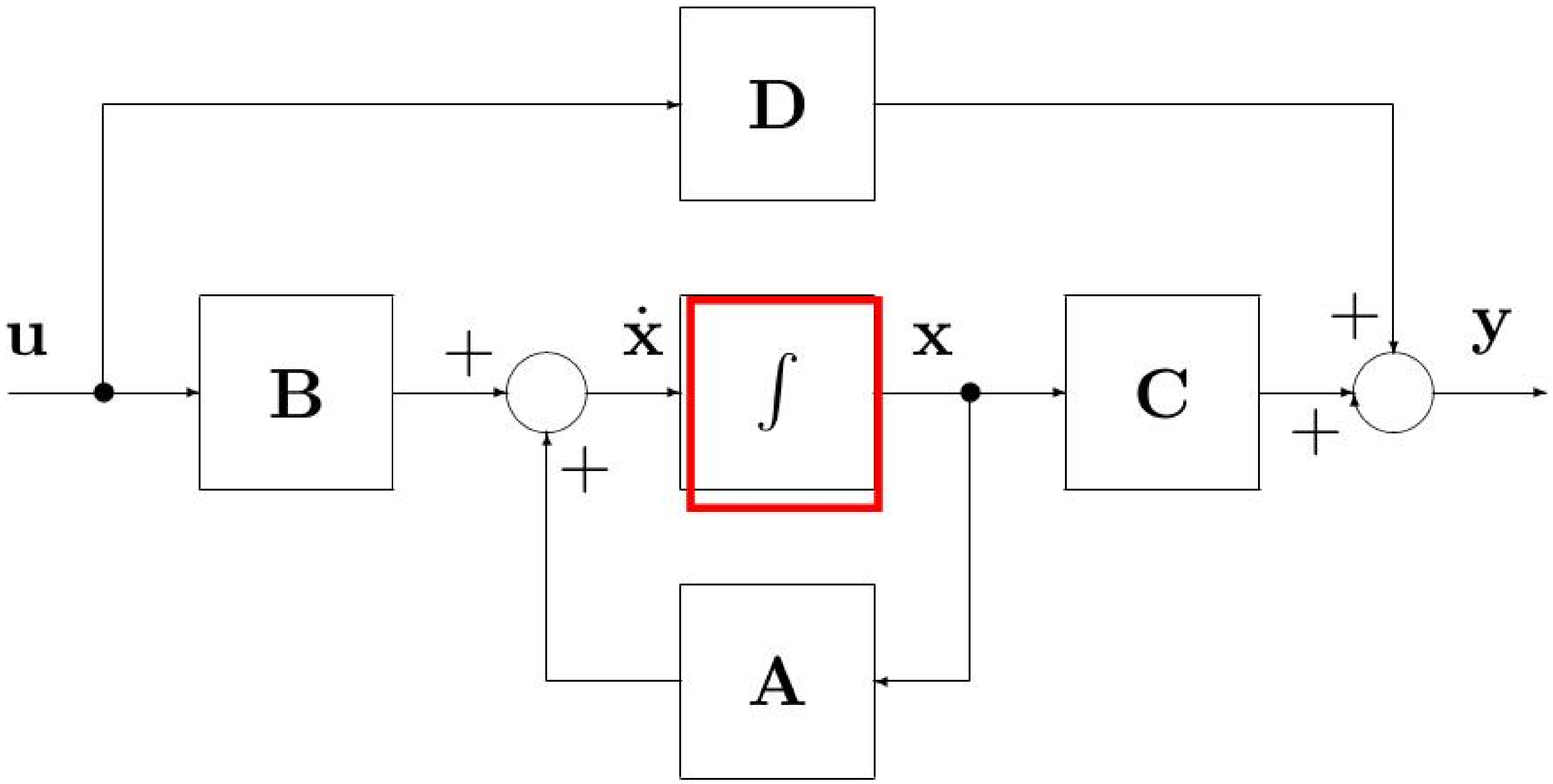
Having vectors of variables, $\mathbf{G}(s)$ is a matrix of functions of s

Its element $\mathbf{G}_{ij}(s)$ is the Laplace image of the response at the jth output to the Dirac pulse at the ith input (difficult to measure directly)

Descriptions of a linear dynamical system

Internal description: $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$



External description: using the Laplace images $\mathbf{U}(s), \mathbf{Y}(s)$ of $\mathbf{u}(t), \mathbf{y}(t)$ (with zero initial conditions)

$\mathbf{Y}(s) = \mathbf{G}(s)\,\mathbf{U}(s)$ where $\mathbf{G}(s)$ is the **transfer function** of the system

Having vectors of variables, $\mathbf{G}(s)$ is a matrix of functions of s

Its element $\mathbf{G}_{ij}(s)$ is the Laplace image of the response at the jth output to the Dirac pulse at the ith input (difficult to measure directly)

How to derive the external description from the internal one?

Under zero initial conditions ($\mathbf{I}$ denotes the unit matrix):

$$\begin{aligned} s\,\mathbf{X}(s) &= \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s) \\ \mathbf{Y}(s) &= \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s) \end{aligned}$$

$$sX(s) - AX(s) = BU(s)$$

$$\begin{aligned} \mathbf{X}(s) &= (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s) \\ \mathbf{Y}(s) &= \underbrace{\left(\mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(s)}\,\mathbf{U}(s) \end{aligned}$$

$\det(s\,\mathbf{I} - \mathbf{A}) = 0$ is the **characteristic equation** of the system
Its solutions (in variable s) are **eigenvalues**
Each of them corresponds to one exponential component of the solution
The system is stable iff all characteristic numbers have negative real parts
(this can be tested without finding the characteristic numbers, which is a difficult problem)

How to derive the internal description from the external one?
This is not unique, there are many standardized methods whose choice depends on the hardware implementation

---

How to derive the external description from the internal one?

Under zero initial conditions ($\mathbf{I}$ denotes the unit matrix):

$$
\begin{aligned}
s\,\mathbf{X}(s) &= \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s) \\
\mathbf{Y}(s) &= \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s)
\end{aligned}
$$

sX(s) - AX(s) = BU(s)

$$
\begin{aligned}
\mathbf{X}(s) &= (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s) \\
\mathbf{Y}(s) &= \underbrace{\left(\mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(s)}\,\mathbf{U}(s)
\end{aligned}
$$

$\det(s\,\mathbf{I} - \mathbf{A}) = 0$ is the **characteristic equation** of the system
Its solutions (in variable s) are **eigenvalues**
Each of them corresponds to one exponential component of the solution
The system is stable iff all characteristic numbers have negative real parts
(this can be tested without finding the characteristic numbers, which is a difficult problem)

How to derive the internal description from the external one?

This is not unique, there are many standardized methods whose choice depends on the hardware implementation

Non-stability of the cartpole

$$s\,\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 & 0 & 0 \\ -\frac{g}{\ell} & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 0 & 0 & 0 & s \end{bmatrix}$$

$$\det(s\,\mathbf{I} - \mathbf{A}) = \frac{1}{\ell}\left(s^2\ell - g\right)s^2$$

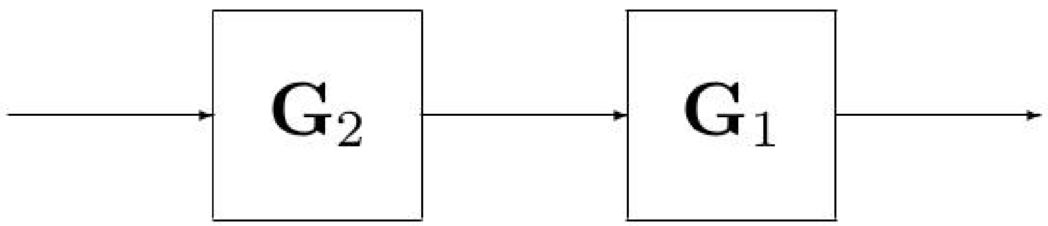
This polynomial has roots 0 (double) and $\pm\sqrt{\frac{g}{\ell}}$
All of them are real, one positive (causing non-stability), one zero (at the boundary of stability region) and one negative (corresponding to a stable component)

$$(s\,\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{-\ell s}{g-s^2\ell} & \frac{-\ell}{g-s^2\ell} & 0 & 0 \\ \frac{-g}{g-s^2\ell} & \frac{-\ell s}{g-s^2\ell} & 0 & 0 \\ 0 & 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & 0 & \frac{1}{s} \end{bmatrix}$$

$$\mathbf{G}(s) = \mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} = \begin{bmatrix} \frac{1}{g-s^2\ell} \\ \frac{1}{s^2} + \frac{\ell}{g-s^2\ell} \end{bmatrix}$$

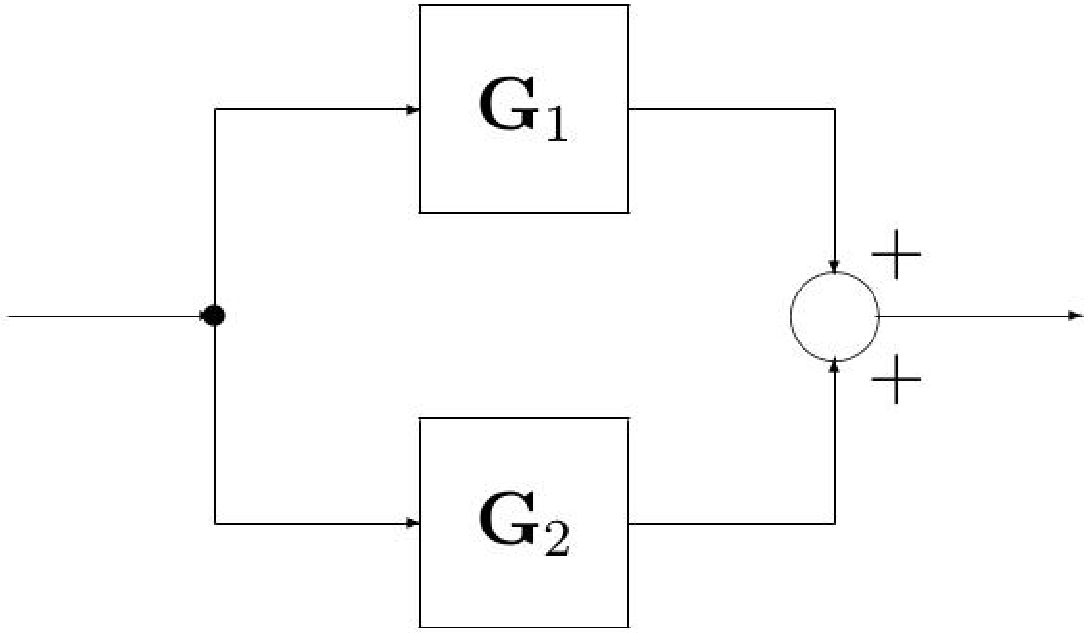
Connections of dynamical systems

Series connection



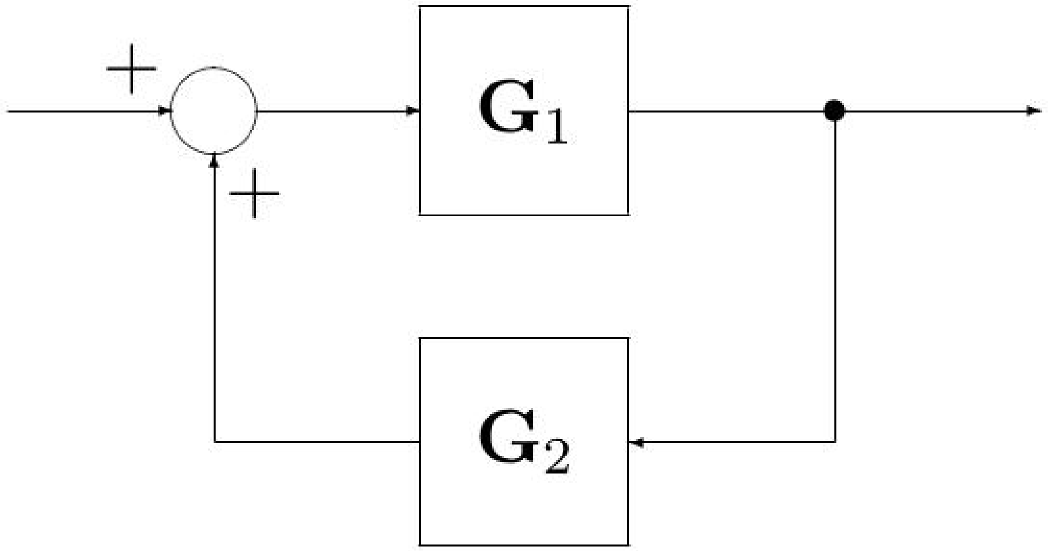
$$\mathbf{G}(s) = \mathbf{G}_1(s)\,\mathbf{G}_2(s)$$

Parallel connection



$$\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$$

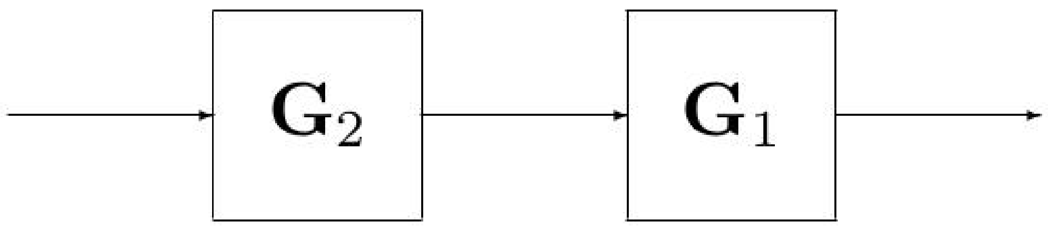
Feedback connection



$$\mathbf{G}(s) = \big(\mathbf{I} - \mathbf{G}_2(s)\,\mathbf{G}_1(s)\big)^{-1}\,\mathbf{G}_1(s)$$

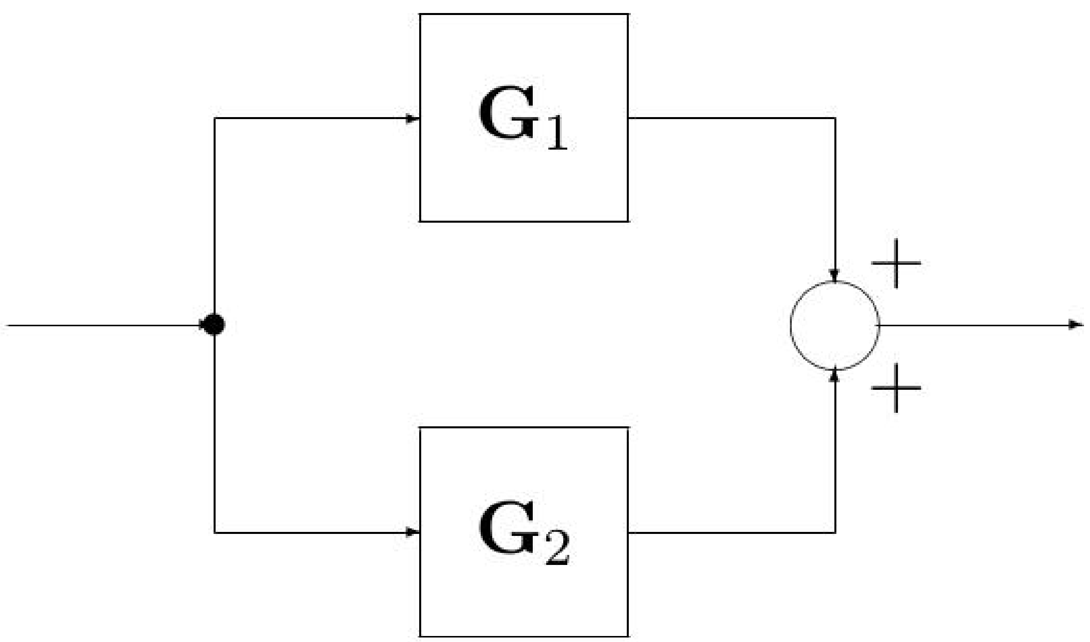
Connections of dynamical systems

Series connection



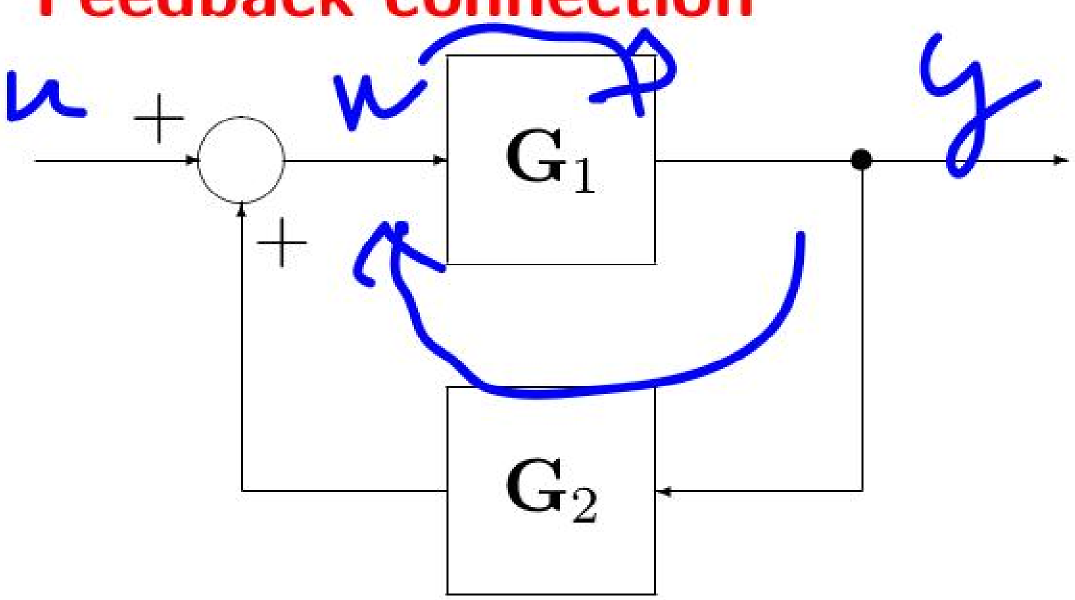
$$\mathbf{G}(s) = \mathbf{G}_1(s)\,\mathbf{G}_2(s)$$

Parallel connection



$$\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$$

Feedback connection



$w = u + G_2 y = u + G_2 G_1 w$

$y = G_1 w$

$(I - G_2 G_1) w = u$

$w = (I - G_2 G_1)^{-1} u \;\big|\; y = G_1 w$

$$\mathbf{G}(s) = \big(\mathbf{I} - \mathbf{G}_2(s)\,\mathbf{G}_1(s)\big)^{-1}\,\mathbf{G}_1(s)$$

Exceptions: Cancellation of roots

Example of a series connection:

$$\begin{aligned}
\mathbf{G}(s) &= \mathbf{G}_1(s)\,\mathbf{G}_2(s) \\
\mathbf{G}_1(s) &= \frac{1}{s+a} \\
\mathbf{G}_2(s) &= \frac{s+a}{s+b} \\
\mathbf{G}(s) &= \frac{1}{s+b}
\end{aligned}$$

The factor $s+a$ has been cancelled and does not occur in the external description, although it corresponds to some internal part;
Moreover, for $a<0$ it is unstable, while the stability of the external description depends only on b

Exceptions: Cancellation of roots

Example of a series connection:

$$
\begin{aligned}
\mathbf{G}(s) &= \mathbf{G}_1(s)\,\mathbf{G}_2(s) \\
\mathbf{G}_1(s) &= \frac{1}{s+a} \\
\mathbf{G}_2(s) &= \frac{s+a}{s+b} \\
\mathbf{G}(s) &= \frac{1}{s+b}
\end{aligned}
$$

The factor $s+a$ has been cancelled and does not occur in the external description, although it corresponds to some internal part;
Moreover, for $a<0$ it is unstable, while the stability of the external description depends only on b

Exceptions: Cancellation of roots

Example of a series connection:

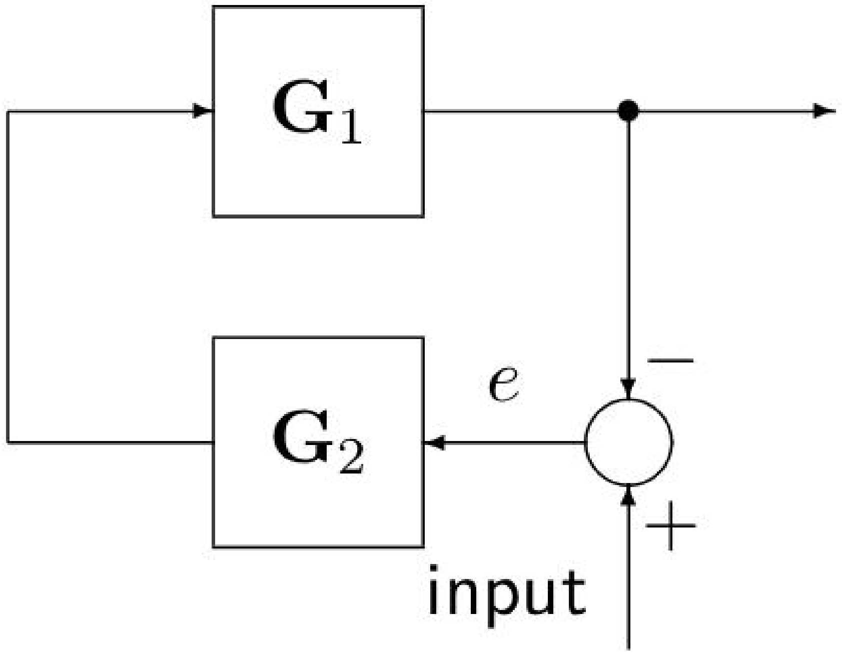
$$\mathbf{G}(s) = \mathbf{G}_1(s)\,\mathbf{G}_2(s)$$
$$\mathbf{G}_1(s) = \frac{1}{s+a}$$
$$\mathbf{G}_2(s) = \frac{s+a}{s+b}$$
$$\mathbf{G}(s) = \frac{1}{s+b}$$

The factor $s+a$ has been cancelled and does not occur in the external description, although it corresponds to some internal part;
Moreover, for $a < 0$ it is unstable, while the stability of the external description depends only on b

This part has a state variable which is not controllable, neither observable

Classical controller design

We usually put the controller in the feedback loop



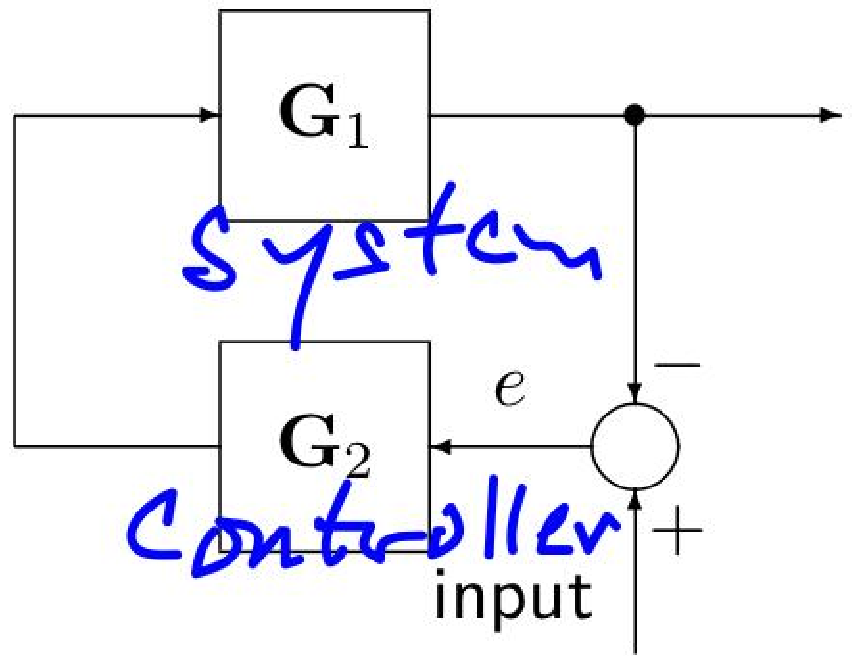
$$\mathbf{G}(s) = \big(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\big)^{-1}\,\mathbf{G}_2(s)\,\mathbf{G}_1(s)$$

The stability of the whole loop is influenced by the factor $\big(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\big)^{-1}$
where $\mathbf{G}_1(s)$ (the controlled system) is given
and $\mathbf{G}_2(s)$ (the controller) can be chosen almost arbitrarily

Using the above analysis, we can decide the stability of the loop with the proposed controller (difficult)

Classical controller design

We usually put the controller in the feedback loop



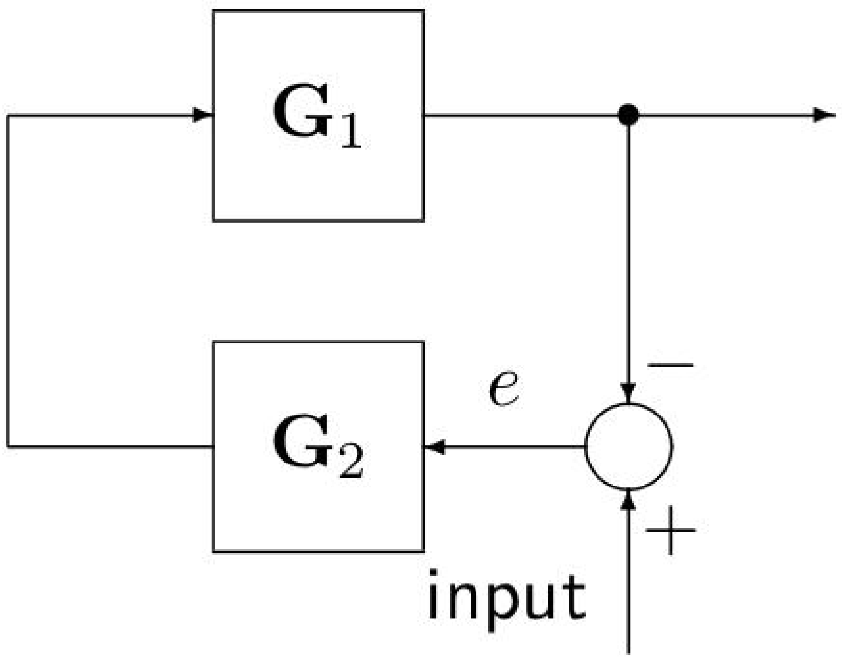
$$\mathbf{G}(s) = \left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}\mathbf{G}_2(s)\,\mathbf{G}_1(s)$$

The stability of the whole loop is influenced by the factor $\left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}$
where $\mathbf{G}_1(s)$ (the controlled system) is given
and $\mathbf{G}_2(s)$ (the controller) can be chosen almost arbitrarily

Using the above analysis, we can decide the stability of the loop with the proposed controller (difficult)

Classical controller design

We usually put the controller in the feedback loop



$$\mathbf{G}(s) = \left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}\mathbf{G}_2(s)\,\mathbf{G}_1(s)$$

The stability of the whole loop is influenced by the factor $\left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}$
where $\mathbf{G}_1(s)$ (the controlled system) is given
and $\mathbf{G}_2(s)$ (the controller) can be chosen almost arbitrarily

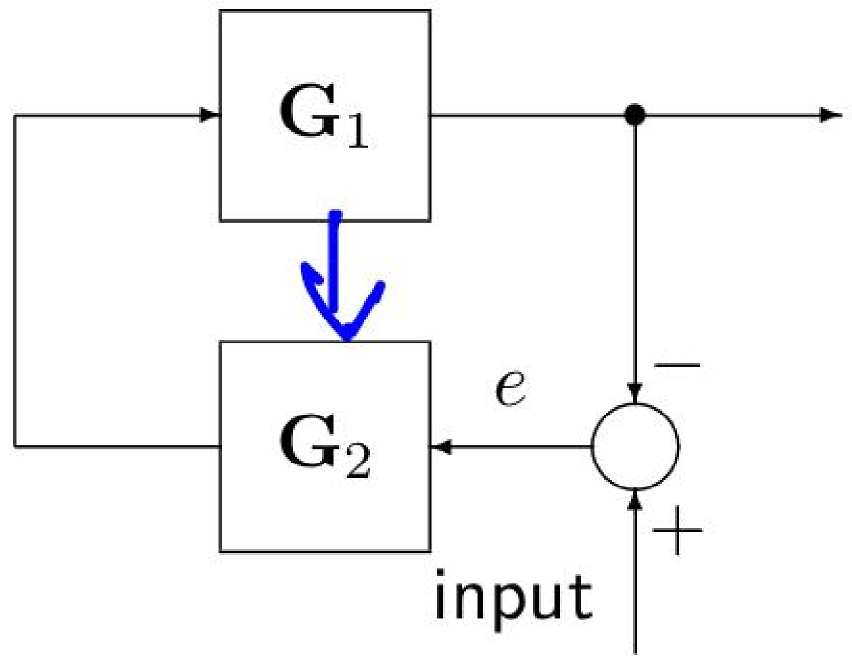
Using the above analysis, we can decide the stability of the loop with the proposed controller (difficult)

The task is easier if we have more information than the output of the controlled system, in particular if we can measure the **states**; then a feedback from states allows – in its extreme (theoretical) form – to achieve arbitrary dynamics of the control loop

Generally, the more information we have the better the control behaviour can be

Classical controller design

We usually put the controller in the feedback loop



$$\mathbf{G}(s) = \left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}\mathbf{G}_2(s)\,\mathbf{G}_1(s)$$

The stability of the whole loop is influenced by the factor $\left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}$
where $\mathbf{G}_1(s)$ (the controlled system) is given
and $\mathbf{G}_2(s)$ (the controller) can be chosen almost arbitrarily

Using the above analysis, we can decide the stability of the loop with the proposed controller (difficult)

The task is easier if we have more information than the output of the controlled system, in particular if we can measure the **states**; then a feedback from states allows – in its extreme (theoretical) form – to achieve arbitrary dynamics of the control loop

Generally, the more information we have the better the control behaviour can be