Fuzzy implication

is any operation $\dot{\underset{\cdot}{\to}} : [0,1]^2 \to [0,1]$ which coincides with the classical implication on $\{0,1\}^2$.
We would like to satisfy the following properties, but we do not require them as axioms:

$$\alpha \dot{\underset{\cdot}{\to}} \beta = 1 \Leftarrow \alpha \le \beta, \tag{I1a}$$

$$\alpha \dot{\underset{\cdot}{\to}} \beta = 1 \Rightarrow \alpha \le \beta, \tag{I1b}$$

$$1 \dot{\underset{\cdot}{\to}} \beta = \beta, \tag{I2}$$

$\dot{\underset{\cdot}{\to}}$ is nonincreasing in the first argument and nondecreasing in the second, (I3)

$$\alpha \dot{\underset{\cdot}{\to}} \beta = \underset{S}{\neg} \beta \dot{\underset{\cdot}{\to}} \underset{S}{\neg} \alpha, \tag{I4}$$

$$\alpha \dot{\underset{\cdot}{\to}} (\beta \dot{\underset{\cdot}{\to}} \gamma) = \beta \dot{\underset{\cdot}{\to}} (\alpha \dot{\underset{\cdot}{\to}} \gamma), \tag{I5}$$

continuity. (I6)

R-implication (residuated fuzzy implication, residuum)

is an operation

$$\alpha \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup\{\gamma : \ \alpha \underset{\cdot}{\wedge} \gamma \leq \beta\} \tag{RI}$$

where $\underset{\cdot}{\wedge}$ is a fuzzy conjunction

(if $\underset{\cdot}{\wedge}$ is continuous, we may take the maximum instead of the supremum)

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where $\underset{\cdot}{\wedge}$ is a fuzzy conjunction

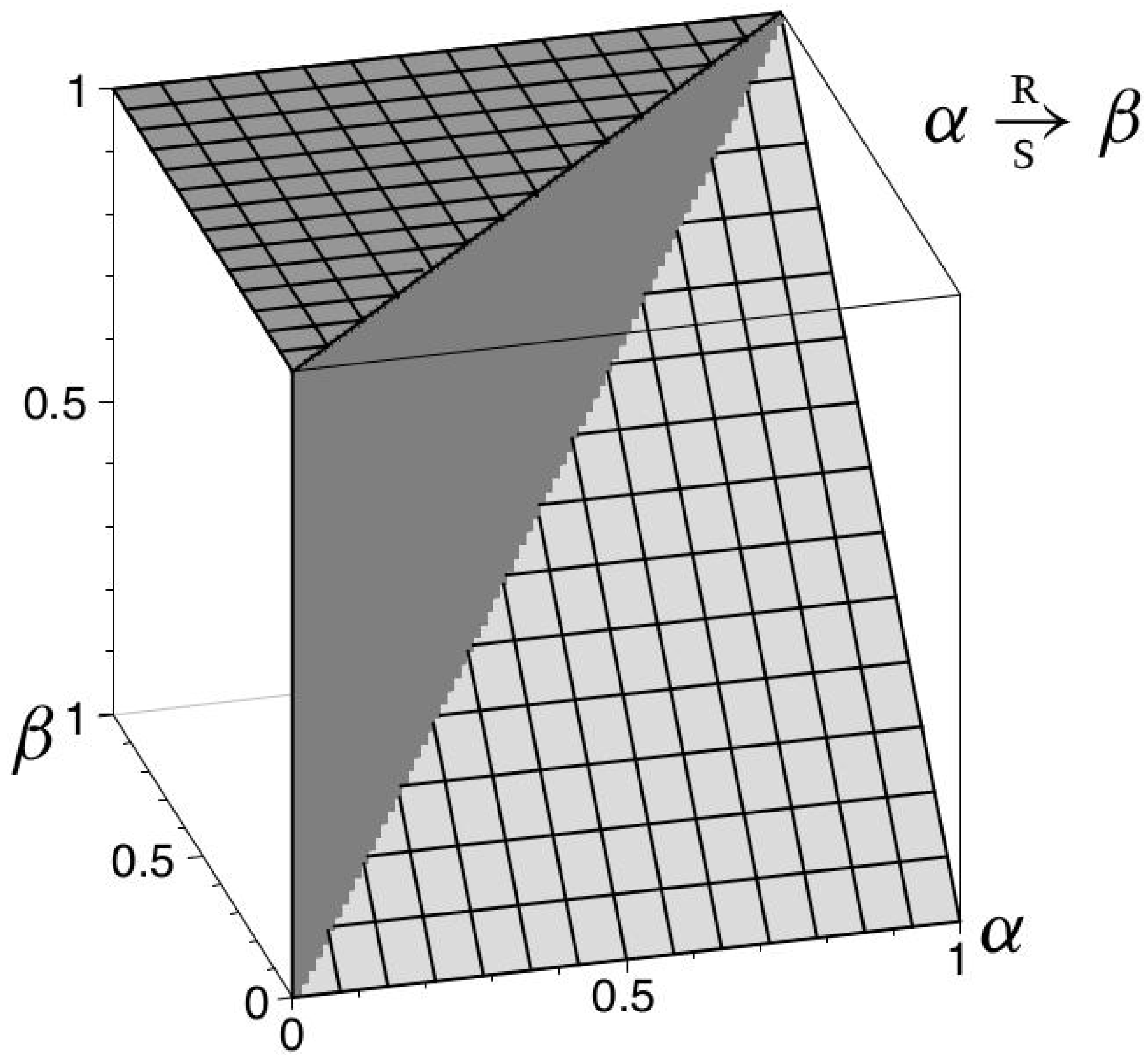
(if $\underset{\cdot}{\wedge}$ is continuous, we may take the maximum instead of the supremum)

Examples of R-implications

- From the standard conjunction $\underset{S}{\wedge}$ we obtain the **Gödel implication**

$$\alpha \overset{R}{\underset{S}{\to}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ \beta & \text{otherwise.} \end{cases}$$

It is piecewise linear and continuous except for the points (α, α), $\alpha < 1$.

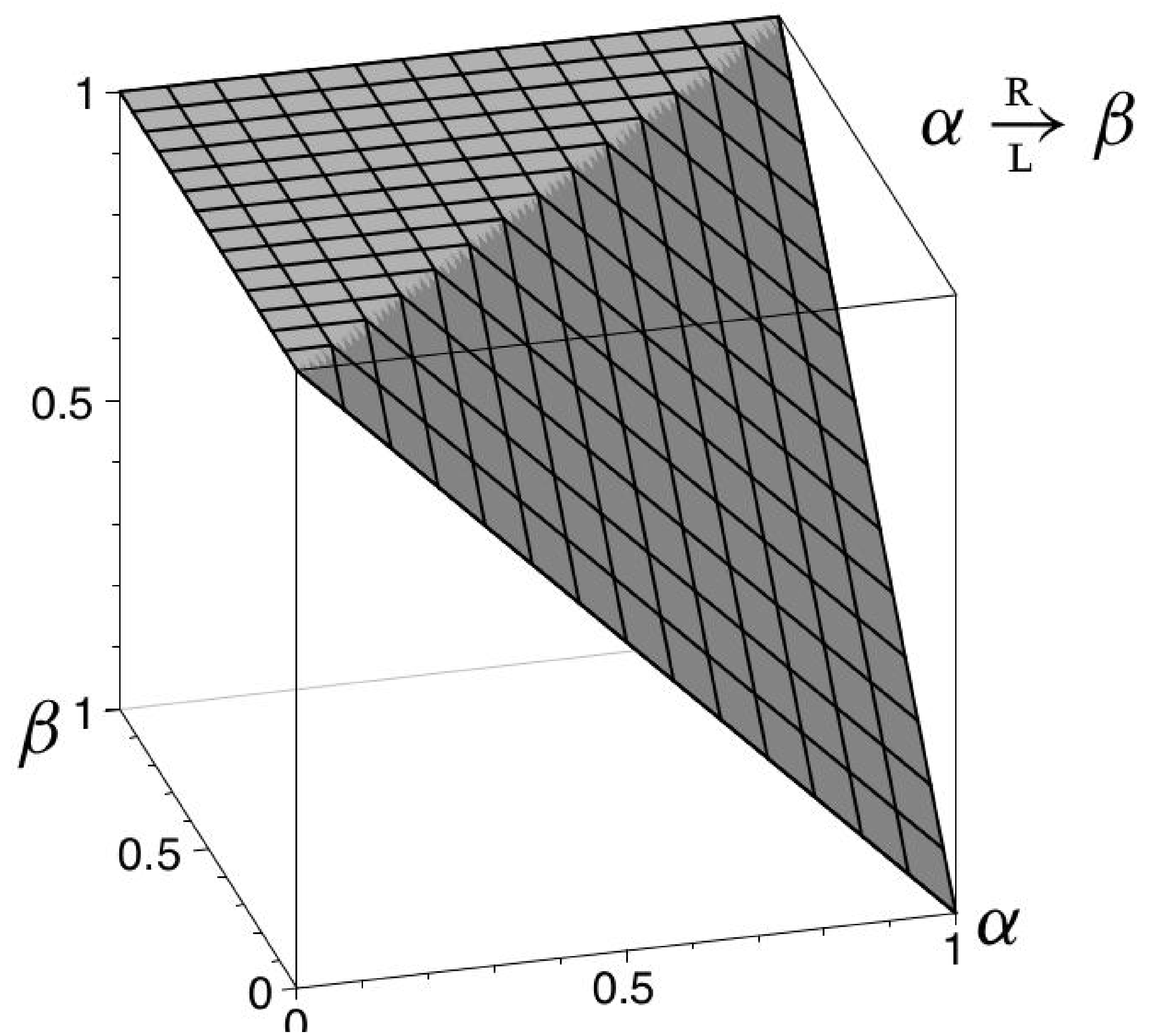


Examples of R-implications

- From the Łukasiewicz conjunction $\underset{\mathrm{L}}{\wedge}$ we obtain the **Łukasiewicz implication**

$$\alpha \overset{\mathrm{R}}{\underset{\mathrm{L}}{\rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ 1 - \alpha + \beta & \text{otherwise.} \end{cases}$$

It is piecewise linear and continuous.

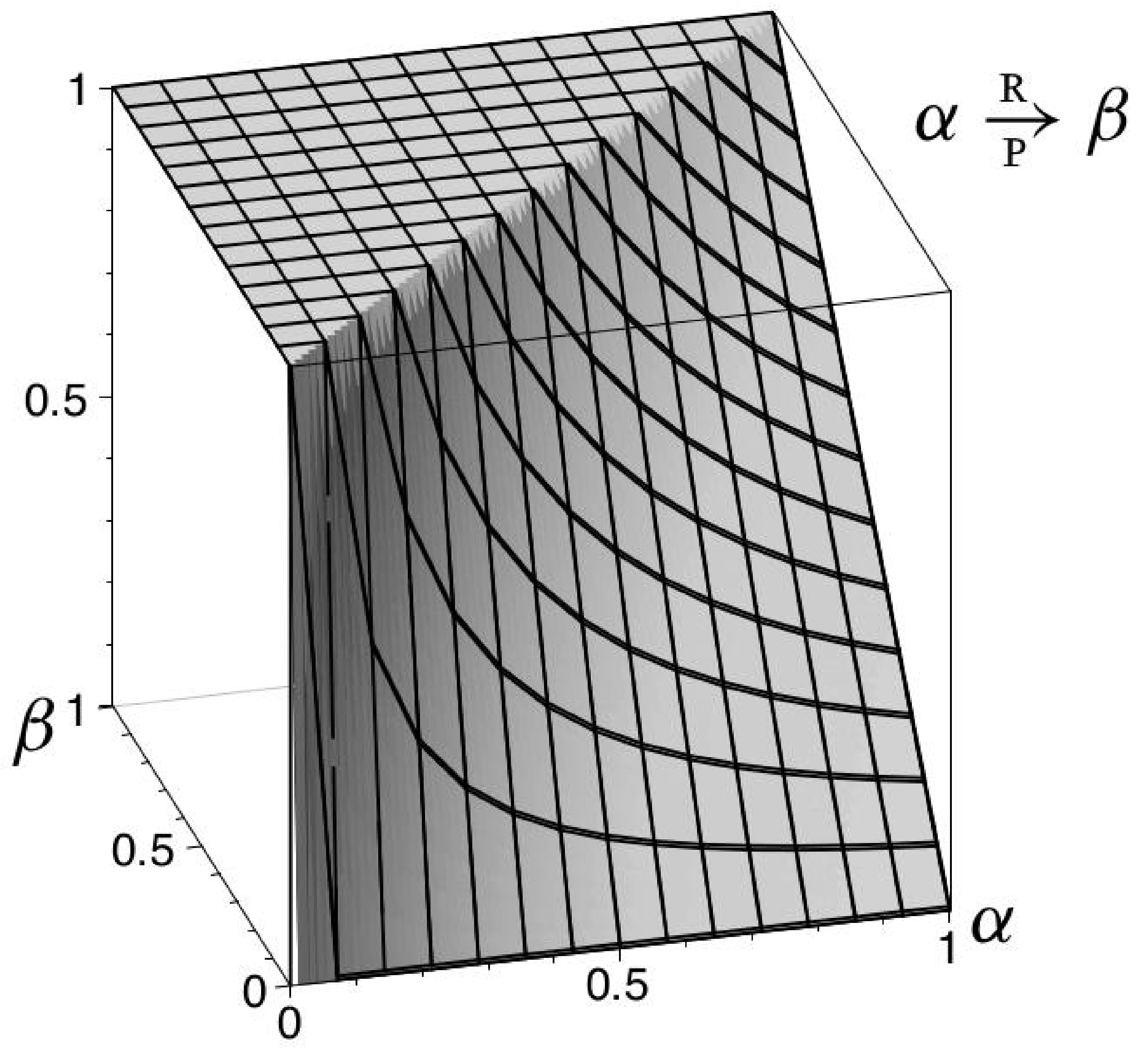


Examples of R-implications

- From the product conjunction $\underset{\mathrm{P}}{\wedge}$ we obtain the **Goguen** (also **Gaines**) **implication**

$$\alpha \overset{\mathrm{R}}{\underset{\mathrm{P}}{\rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ \frac{\beta}{\alpha} & \text{otherwise.} \end{cases}$$

It has one point of discontinuity, $(0, 0)$.



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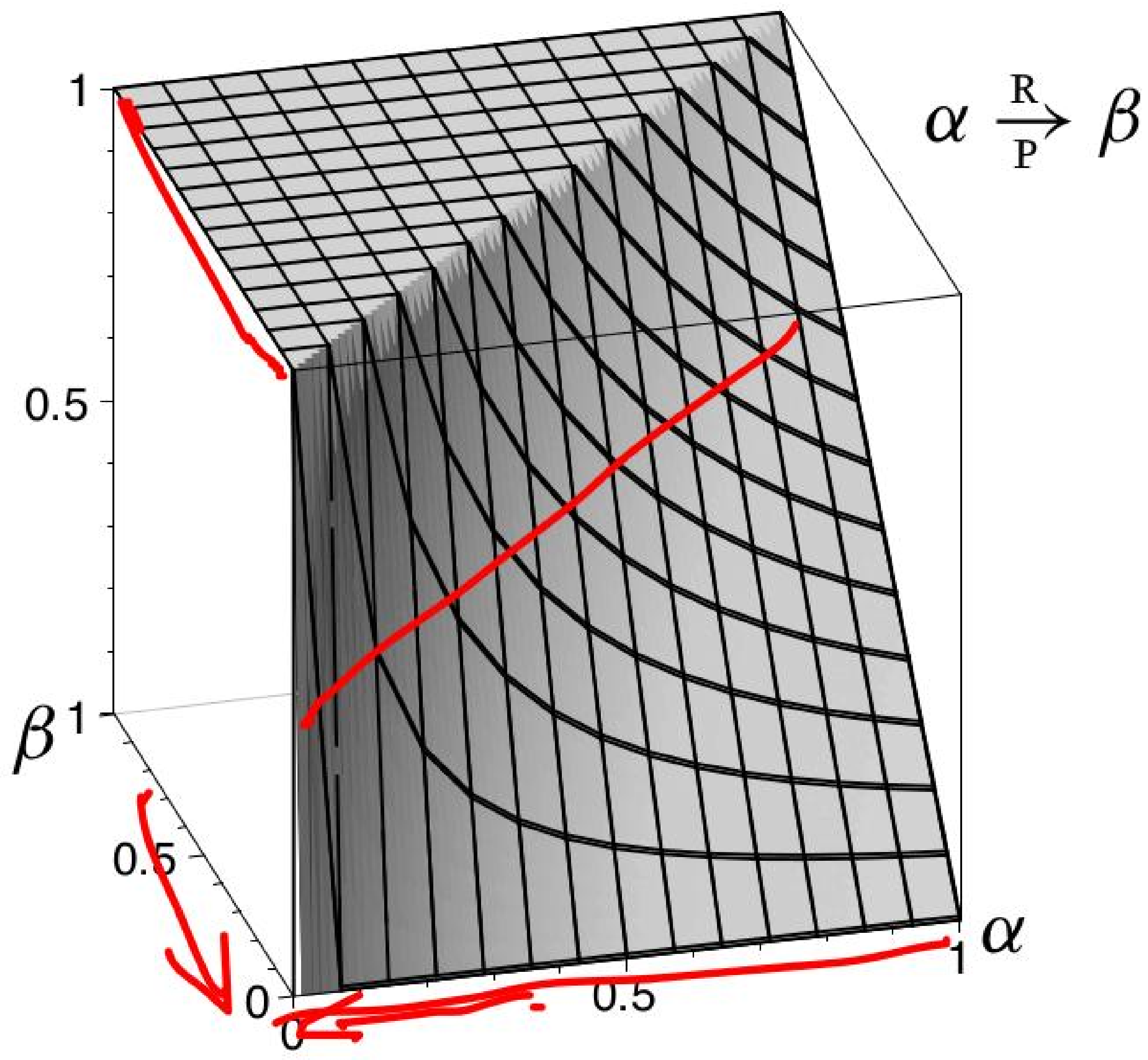

Properties of R-implications

Theorem: Let $\underset{\cdot}{\wedge}$ be a continuous fuzzy conjunction. Then the R-implication $\overset{\mathrm{R}}{\underset{\cdot}{\to}}$ satisfies (I1a), (I1b), (I2), (I3).

Proof: $\alpha \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup \Gamma(\alpha, \beta)$, where
$\Gamma(\alpha, \beta) = \{\gamma : \alpha \underset{\cdot}{\wedge} \gamma \leq \beta\}$ is an interval containing zero. (Moreover, due to the continuity of $\underset{\cdot}{\wedge}$ the interval is closed.)

(I1a) If $\alpha \leq \beta$, then $\Gamma(\alpha, \beta) = [0, 1]$, $\sup \Gamma(\alpha, \beta) = 1$.

(I1b) If $\alpha > \beta$, then $1 \notin \Gamma(\alpha, \beta)$, $\sup \Gamma(\alpha, \beta) < 1$ (from the closedness of $\Gamma(\alpha, \beta)$).

(I2): $1 \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup\{\gamma : \gamma \leq \beta\} = \beta$.

(I3): When α increases, $\Gamma(\alpha, \beta)$ does not increase.
When β increases, $\Gamma(\alpha, \beta)$ does not decrease.

Theorem: A residuated fuzzy implication induced by a **continuous** fuzzy conjunction $\underset{\cdot}{\wedge}$ is continuous iff $\underset{\cdot}{\wedge}$ is nilpotent.

Properties of R-implications

Theorem: Let $\dot{\wedge}$ be a continuous fuzzy conjunction. Then the R-implication $\overset{R}{\underset{\cdot}{\to}}$ satisfies (I1a), (I1b), (I2), (I3).

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S-implication

is an operation

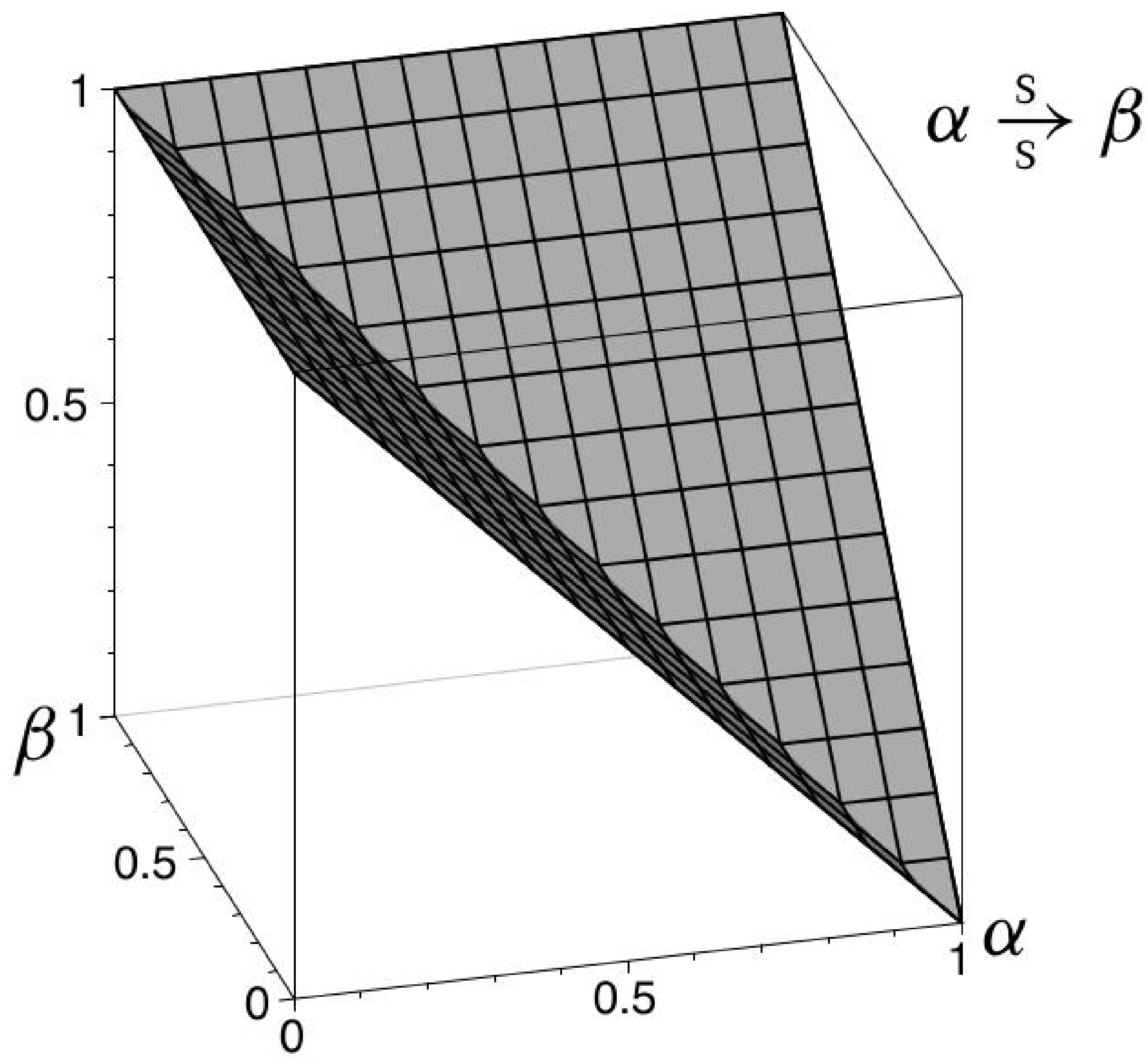
$$\alpha \overset{\mathrm{S}}{\underset{\cdot}{\rightarrow}} \beta = \underset{\mathrm{S}}{\neg}\, \alpha \,\dot{\vee}\, \beta \qquad \text{(SI)}$$

where $\dot{\vee}$ is a fuzzy disjunction

Examples of S-implications

- From the standard disjunction we obtain the **Kleene–Dienes** implication

$$\alpha \underset{\mathrm{S}}{\overset{\mathrm{S}}{\to}} \beta = \max(1-\alpha, \beta).$$



Examples of S-implications

- From the Łukasiewicz disjunction we obtain the **Łukasiewicz** implication $\overset{\mathrm{S}}{\underset{\mathrm{L}}{\rightarrow}}$ which coincides with the Łukasiewicz residuated implication $\overset{\mathrm{R}}{\underset{\mathrm{L}}{\rightarrow}}$.

Among all fuzzy implications studied here, only residuated implications induced by nilpotent fuzzy conjunctions (e.g., the Łukasiewicz implication) satisfy all properties (I1a),(I1b),(I2)–(I6).

Fuzzy biimplication (equivalence)

is an operation $\underset{\cdot}{\dot{\leftrightarrow}}$ usually defined by

$$\alpha \underset{\cdot}{\dot{\leftrightarrow}} \beta = (\alpha \underset{\cdot}{\dot{\rightarrow}} \beta) \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\dot{\rightarrow}} \alpha),$$

where $\underset{\cdot}{\dot{\rightarrow}}$ is a fuzzy implication and $\underset{\cdot}{\wedge}$ is a fuzzy conjunction
(biimplications are distinguished by the same indices as the respective fuzzy implications)

If $\underset{\cdot}{\dot{\rightarrow}}$ satisfies (I1a) (e.g., for a residuated implication), at least one of the brackets equals 1, hence the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$ is irrelevant.

Example: **Łukasiewicz** biimplication: $\alpha \overset{\mathrm{R}}{\underset{\mathrm{L}}{\leftrightarrow}} \beta = 1 - |\alpha - \beta|.$

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Classical relations

A **binary relation** is an $R \subseteq X \times Y$

Inverse relation to R: $R^{-1} \subseteq Y \times X$:

$$R^{-1} = \{(y, x) \in Y \times X : \ (x, y) \in R\}$$

The **composition** of relations $R \subseteq X \times Y$, $S \subseteq Y \times Z$ is $R \circ S \subseteq X \times Z$:

$$R \circ S = \Big\{(x, z) \in X \times Z : \ \big(\exists y \in Y : (x, y) \in R, (y, z) \in S\big)\Big\}$$

Using membership functions:

$$\mu_R : X \times Y \to \{0, 1\}$$

$$\mu_{R^{-1}}(y, x) = \mu_R(x, y)$$

$$\mu_{R \circ S}(x, z) = \max_{y \in Y}\big(\mu_R(x, y) \wedge \mu_S(y, z)\big)$$

Fuzzy relations

A **fuzzy relation** is $R \in \mathcal{F}(X \times Y)$, $\mu_R : X \times Y \to [0, 1]$

The **inverse relation** to R is $R^{-1} \in \mathcal{F}(Y \times X)$:

$$\forall x \in X \ \forall y \in Y : \mu_{R^{-1}}(y, x) = \mu_R(x, y)$$

The $\dot{}$**-composition** of relations $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ is $R \underset{\cdot}{\circ} S \in \mathcal{F}(X \times Z)$:

$$\mu_{R \underset{\cdot}{\circ} S}(x, z) = \sup_{y \in Y} \left(\mu_R(x, y) \underset{\cdot}{\wedge} \mu_S(y, z)\right)$$

Theorem The inversion of fuzzy relations is cut-consistent.

Theorem If Y is a finite set, then the standard composition of fuzzy relations $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ is cut-consistent.

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Special crisp relations

$R \subseteq X \times X$ can be:

- **an equality**: $E = \{(x,x) : x \in X\}$,
- **reflexive**: $\forall x \in X : (x,x) \in R$, i.e., $E \subseteq R$,
- **symmetric**: $(x,y) \in R \Rightarrow (y,x) \in R$, i.e., $R = R^{-1}$,
- **antisymmetric**: $\big((x,y) \in R\big) \wedge \big((y,x) \in R\big) \Rightarrow x = y$, i.e., $R \cap R^{-1} \subseteq E$,
- **transitive**: $\big((x,y) \in R\big) \wedge \big((y,z) \in R\big) \Rightarrow (x,z) \in R$, i.e., $R \circ R \subseteq R$,
- **a partial order**: antisymmetric, reflexive, and transitive,
- **an equivalence**: symmetric, reflexive, and transitive.

The membership function of the equality relation, $E \subseteq X \times X$, is the **Kronecker delta**:

$$\mu_E(x,y) = \delta(x,y) = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{for } x \neq y. \end{cases}$$

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Special fuzzy relations

A fuzzy relation $R \in \mathcal{F}(X \times X)$ can be:

- **reflexive**: $E \subseteq R$,
- **symmetric**: $R = R^{-1}$,
- **$\cdot$-antisymmetric**: $R \underset{\cdot}{\cap} R^{-1} \subseteq E$,
- **$\cdot$-transitive**: $R \underset{\cdot}{\circ} R \subseteq R$,
- **a $\cdot$-partial order**: $\cdot$-antisymmetric, reflexive, and $\cdot$-transitive,
- **an $\cdot$-equivalence**: symmetric, reflexive, and $\cdot$-transitive.

The last four terms depend on the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$.