Deduction theorem in classical logic

$\mathcal{T}$ ... theory

$A, B$ ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \rightarrow B$

Proof \hspace{1cm} $\Leftarrow$:

//BEGIN of proof of $\mathcal{T} \vdash A \rightarrow B$

: 

$D_{i-1} = A \rightarrow B$

//END of proof of $\mathcal{T} \vdash A \rightarrow B$

SA : $D_i = A$

MP($D_i, D_{i-1}$) : $D_{i+1} = B$
Deduction theorem in classical logic

$\mathcal{T}$ ... theory

$A, B$ ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \rightarrow B$

Proof

$\Leftarrow$:

//BEGIN of proof of $\mathcal{T} \vdash A \rightarrow B$

$D_{i-1} = A \rightarrow B$

//END of proof of $\mathcal{T} \vdash A \rightarrow B$

SA : $D_i = A$

MP($D_i, D_{i-1}$) : $D_{i+1} = B$
⇒: Proof by contradiction:
Suppose that there is a formula $B$ such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \not\vdash A \to B$. 
⇒: Proof by contradiction:
Suppose that there is a formula $B$ such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \not\vdash A \rightarrow B$.

1. $B$ is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

\[
D_1 = B \\
RI(D_1) : D_2 = A \rightarrow B
\]

hence $\mathcal{T} \vdash A \rightarrow B$. 

⇒: Proof by contradiction:
Suppose that there is a formula $B$ such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \not\vdash A \rightarrow B$.

1. $B$ is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

$$D_1 = B$$

$$RI(D_1): \quad D_2 = A \rightarrow B$$

hence $\mathcal{T} \vdash A \rightarrow B$.

2. $B \neq A$ because $\mathcal{T} \vdash A \rightarrow A$. 
3. \( B \) is obtained by deduction in the proof of \( \mathcal{T} \cup \{A\} \vdash B \).

WLOG, we choose for \( B \) a formula with the shortest possible proof; its shortest proof must be of the following form:

\[
\vdots \\
D_i \\
\vdots \\
D_j = D_i \rightarrow B \\
\vdots \\
MP(D_i, D_j) : D_m = B
\]

for \( i < j < m \) or \( j < i < m \).
3. $B$ is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.

WLOG, we choose for $B$ a formula with the shortest possible proof; its shortest proof must be of the following form:

\[ \vdots \]
\[ D_i \]
\[ \vdots \]
\[ D_j = D_i \rightarrow B \]
\[ \vdots \]
\[ \text{MP}(D_i, D_j) : D_m = B \]

for $i < j < m$ or $j < i < m$.

The proofs of $\mathcal{T} \cup \{A\} \vdash D_i$, $\mathcal{T} \cup \{A\} \vdash D_j$ are of lengths $< m$, therefore

$\mathcal{T} \vdash A \rightarrow D_i$

$\mathcal{T} \vdash A \rightarrow D_j = A \rightarrow (D_i \rightarrow B)$
Proof of $\mathcal{T} \vdash A \rightarrow B$:

//BEGIN of proof of $\mathcal{T} \vdash A \rightarrow D_i$

$\vdots$

$D_k = A \rightarrow D_i$

//END of proof of $\mathcal{T} \vdash A \rightarrow D_i$

//BEGIN of proof of $\mathcal{T} \vdash A \rightarrow D_j$

$\vdots$

$D_n = A \rightarrow \overbrace{(D_i \rightarrow B)}^{D_j}$

//END of proof of $\mathcal{T} \vdash A \rightarrow D_j$

(C2) $B := D_i, C := B$:

$D_{n+1} = (A \rightarrow (D_i \rightarrow B)) \rightarrow ((A \rightarrow D_i) \rightarrow (A \rightarrow B))$

MP($D_n, D_{n+1}$):

$D_{n+2} = (A \rightarrow D_i) \rightarrow (A \rightarrow B)$

MP($D_k, D_{n+2}$):

$D_{n+3} = A \rightarrow B$
Corollary Cor2 \[ A \vdash A \lor B \] for all \( A, B \)

\[ A \vdash \neg A \rightarrow B = A \lor B \]

\( \Downarrow (\text{DT}) \)

\[ \text{ALL}(A) : \quad \{ A, \neg A \} \vdash B \]

\( \Rightarrow \) we can add a deduction rule \( \frac{A}{A \lor B} \) (and \( \frac{B}{A \lor B} \) was already proved in Cl9)
Corollary Cor3  
\[ A \vdash \neg \neg A, \quad \vdash A \rightarrow \neg \neg A \text{ for all } A \]

\[ \vdash A \rightarrow (\neg A \rightarrow 0) \]

\[ \vdash (DT) \]

\[ A \vdash \neg A \rightarrow 0 \]

\[ \vdash (DT) \]

\[ \text{ALL}(A) : \quad \{A, \neg A\} \vdash 0 \]

Corollary Cor4  
\[ \neg \neg A \vdash A, \quad \vdash \neg \neg A \rightarrow A \text{ for all } A \]

Cor3,  
\[ A := \neg A : \quad D_1 = \neg A \rightarrow \neg \neg A \]

(C3)  
\[ B := \neg \neg A : \quad D_2 := (\neg A \rightarrow \neg \neg A) \rightarrow (\neg A \rightarrow A) \]

MP\((D_1, D_2)\) :  
\[ D_3 = \neg A \rightarrow A \]

Corollary Cor5  
\[ \vdash A \leftrightarrow \neg \neg A \quad (\text{can be added to axioms}) \]

How can we simplify our proofs?

\[ B \leftrightarrow C \vdash (A \rightarrow B) \leftrightarrow (A \rightarrow C) \]

\[ B \leftrightarrow C \vdash (B \rightarrow A) \leftrightarrow (C \rightarrow A) \]
Corollary Cor2 \quad A \vdash A \lor B \text{ for all } A, B

A \vdash \neg A \rightarrow B = A \lor B

\Downarrow \text{(DT)}

\text{ALL}(A) : \quad \{A, \neg A\} \vdash B

\Rightarrow \text{we can add a deduction rule } \frac{A}{A \lor B} \text{ (and } \frac{B}{A \lor B} \text{ was already proved in Cl9)}
Corollary Cor2  

\[ A \vdash A \lor B \text{ for all } A, B \]

\[ A \vdash \neg A \rightarrow B = A \lor B \]

\[ \Downarrow (DT) \]

\[ \text{ALL}(A): \quad \{ A, \neg A \} \vdash B \]

\[ \Rightarrow \text{we can add a deduction rule } \frac{A}{A \lor B} \text{ (and } \frac{B}{A \lor B} \text{ was already proved in Cl9)} \]
**Corollary Cor3**

\[ A \vdash \neg \neg A, \quad \vdash A \rightarrow \neg \neg A \text{ for all } A \]

\[ \vdash A \rightarrow (\neg A \rightarrow 0) \]

\[ \Downarrow \text{(DT)} \]

\[ A \vdash \neg A \rightarrow 0 \]

\[ \Downarrow \text{(DT)} \]

\[ \text{ALL}(A) : \quad \{A, \neg A\} \vdash 0 \]

**Corollary Cor4**

\[ \neg \neg A \vdash A, \quad \vdash \neg \neg A \rightarrow A \text{ for all } A \]

\[ \text{Cor3, } A := \neg A : \quad D_1 = \neg A \rightarrow \neg \neg \neg A \]

\[ (C3) \quad B := \neg \neg A : \quad D_2 := (\neg A \rightarrow \neg \neg \neg A) \rightarrow (\neg \neg A \rightarrow A) \]

\[ \text{MP}(D_1, D_2) : \quad D_3 = \neg \neg A \rightarrow A \]

**Corollary Cor5**

\[ \vdash A \leftrightarrow \neg \neg A \quad \text{(can be added to axioms)} \]

**How can we simplify our proofs?**

\[ B \leftrightarrow C \vdash (A \rightarrow B) \leftrightarrow (A \rightarrow C) \]

\[ B \leftrightarrow C \vdash (B \rightarrow A) \leftrightarrow (C \rightarrow A) \]
**Corollary Cor3**

\[ A \vdash \neg \neg A, \quad \vdash A \rightarrow \neg \neg A \text{ for all } A \]

\[ \vdash A \rightarrow (\neg A \rightarrow 0) \]

\[ \checkmark (\text{DT}) \]

\[ A \vdash \neg A \rightarrow 0 \]

\[ \checkmark (\text{DT}) \]

\[ \text{ALL}(A) : \quad \{A, \neg A\} \vdash 0 \]

**Corollary Cor4**

\[ \neg \neg A \vdash A, \quad \vdash \neg \neg A \rightarrow A \text{ for all } A \]

\[ \text{Cor3, } A := \neg A : \quad D_1 = \neg A \rightarrow \neg \neg A \]

\[ (\text{C3}) B := \neg A : \quad D_2 := (\neg A \rightarrow \neg \neg A) \rightarrow (\neg A \rightarrow A) \]

\[ \text{MP}(D_1, D_2) : \quad D_3 = \neg \neg A \rightarrow A \]

**Corollary Cor5**

\[ \vdash A \leftrightarrow \neg \neg A \quad (\text{can be added to axioms}) \]

**How can we simplify our proofs?**

\[ B \leftrightarrow C \vdash (A \rightarrow B) \leftrightarrow (A \rightarrow C) \]

\[ B \leftrightarrow C \vdash (B \rightarrow A) \leftrightarrow (C \rightarrow A) \]
INTERPLAY OF SYNTAX AND SEMANTICS OF CLASSICAL LOGIC

**Weak soundness**  
Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.

**Strong soundness**  
For any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

**Weak completeness**  
Each tautology is provable, i.e., if $\models A$, then $\vdash A$.

**Strong completeness**  
For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$. 
INTERPLAY OF SYNTAX AND SEMANTICS OF CLASSICAL LOGIC

Weak soundness  
Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.

Strong soundness  
For any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness  
Each tautology is provable, i.e., if $\models A$, then $\vdash A$.

Strong completeness  
For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$. 
BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

\( \mathcal{A} \) ... countable set of propositional variables
\( \mathcal{L} = \{ \rightarrow, 0, \wedge \} \) ... the set of logical connectives:
\( \rightarrow \) ... (binary) implication
0 ... (nullary) false
\( \wedge \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\( \neg A = A \rightarrow 0 \) ... (unary) negation
\( 1 = \neg 0 = 0 \rightarrow 0 \) ... (nullary) true
\( A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) \) ... (binary) equivalence
\( A \ Syn \ B = A \wedge (A \rightarrow B) \)
\( A \ Syn \ B = ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A) \)
no \( A \vee B \) in general
INTERPLAY OF SYNTAX AND SEMANTICS OF CLASSICAL LOGIC

Weak soundness: Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.

Strong soundness: For any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness: Each tautology is provable, i.e., if $\models A$, then $\vdash A$.

Strong completeness: For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$. 
INTERPLAY OF SYNTAX AND SEMANTICS OF CLASSICAL LOGIC

**Weak soundness**
Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.

**Strong soundness**
For any theory $T$, if $T \vdash A$, then $T \models A$.

**Weak completeness**
Each tautology is provable, i.e., if $\models A$, then $\vdash A$.

**Strong completeness**
For any finite theory $T$, if $T \models A$, then $T \vdash A$. 

Is $A$ a tautology? 

C3. Axiom...

deduce...
BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

\( \mathcal{A} \) ... countable set of propositional variables

\( \mathcal{L} = \{\rightarrow, 0, \land\} \) ... the set of logical connectives:

\( \rightarrow \) ... (binary) implication

\( 0 \) ... (nullary) false

\( \land \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:

\( \neg A = A \rightarrow 0 \) ... (unary) negation

\( 1 = \neg 0 = 0 \rightarrow 0 \) ... (nullary) true

\( A \leftrightarrow B = (A \rightarrow B) \land (B \rightarrow A) \) ... (binary) equivalence

\( A \land B = A \land (A \rightarrow B) \)

\( A \lor B = ((A \rightarrow B) \rightarrow B) \lor ((B \rightarrow A) \rightarrow A) \)

no \( A \lor B \) in general
BASIC LOGIC (BL)  
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1
\[ A \] ... countable set of propositional variables
\[ \mathcal{L} = \{ \rightarrow, 0, \wedge \} \] ... the set of logical connectives:
\[ \rightarrow \] ... (binary) implication
\[ 0 \] ... (nullary) false
\[ \wedge \] ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\[ \neg A = A \rightarrow 0 \] ... (unary) negation
\[ 1 = \neg 0 = 0 \rightarrow 0 \] ... (nullary) true
\[ A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) \] ... (binary) equivalence
\[ A \triangleleft B = A \wedge (A \rightarrow B) \]
\[ A \triangledown B = ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A) \]
no \[ A \lor B \] in general
**SEMANTICS OF BASIC LOGIC**

In general: a BL-algebra, here only

**Standard semantics** the set of truth values ... \([0, 1]\)

\(\wedge\) ... continuous fuzzy conjunction \(\wedge\)

\(\rightarrow\) ... residuum \(\rightarrow\) of \(\wedge\)

\(0\) ... \(0\)

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:

\(\neg\) ... \(\neg\), \(\text{where } \neg \alpha = \alpha \rightarrow 0\)

\(1\) ... \(1\)

\(\leftrightarrow\) ... \(\leftrightarrow\), \(\text{where } \alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)\)

\(\wedge_{\overline{S}}\) ... \(\wedge_{\overline{S}} = \text{min}\)

\(\vee_{\overline{S}}\) ... \(\vee_{\overline{S}} = \text{max}\)

**Exercise** Verify that the interpretation of \(\wedge_{\overline{S}}, \vee_{\overline{S}}\) is independent of the choice of the fuzzy conjunction.
SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

**Standard semantics**
the set of truth values ... \([0, 1]\)
\(\land\) ... continuous fuzzy conjunction \(\land\)
\(\rightarrow\) ... residuum \(\rightarrow\) of \(\land\)
0 ... 0

Even the standard semantics **is not unique**, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:
\(-\) ... \(-\), where \(-\alpha = \alpha \rightarrow 0\)
1 ... 1
\(\leftrightarrow\) ... \(\leftrightarrow\), where \(\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)\)
\(\land^S\) ... \(\land^S = \min\)
\(\lor^S\) \(\lor^S = \max\)

**Exercise**
Verify that the interpretation of \(\land^S\), \(\lor^S\) is independent of the choice of the fuzzy conjunction.
BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1
\( \mathcal{A} \) ... countable set of propositional variables
\( \mathcal{L} = \{\rightarrow, 0, \wedge\} \) ... the set of logical connectives:
\( \rightarrow \) ... (binary) implication
0 ... (nulary) false
\( \wedge \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\( \neg A = A \rightarrow 0 \) ... (unary) negation
1 = \( \neg 0 = 0 \rightarrow 0 \) ... (nulary) true
\( A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) \) ... (binary) equivalence
\( A \triangleleft B = A \wedge (A \rightarrow B) \)
\( A \triangledown B = ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A) \)
no \( A \vee B \) in general
BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

\( A \) ... countable set of propositional variables
\( \mathcal{L} = \{ \rightarrow, 0, \wedge \} \) ... the set of logical connectives:
\( \rightarrow \) ... (binary) implication
\( 0 \) ... (nulay) false
\( \wedge \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\( \neg A = A \rightarrow 0 \) ... (unary) negation
\( 1 = \neg 0 = 0 \rightarrow 0 \) ... (nulay) true
\( A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) \) ... (binary) equivalence
\( A \diamondsuit B = A \wedge (A \rightarrow B) \)
\( A \oslash B = ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A) \)
no \( A \lor B \) in general
SEMANTICS OF BASIC LOGIC  

In general: a BL-algebra, here only

**Standard semantics**  
the set of truth values ... \([0, 1]\)

\(\wedge\) ... continuous fuzzy conjunction \(\wedge\)

\(\rightarrow\) ... residuum \(\rightarrow\) of \(\wedge\)

\(0\) ... \(0\)

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:

\(\neg\) ... \(\neg\),  
where \(\neg \alpha = \alpha \rightarrow 0\)

\(1\) ... \(1\)

\(\leftrightarrow\) ... \(\leftrightarrow\),  
where \(\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)\)

\(\land\) ... \(\land\) = \(\text{min}\)

\(\lor\) ... \(\lor\) = \(\text{max}\)

**Exercise**  
Verify that the interpretation of \(\land\), \(\lor\) is independent of the choice of the fuzzy conjunction.
SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

**Standard semantics**
the set of truth values ... \([0, 1]\)

\(\land\) ... continuous fuzzy conjunction \(\land\)

\(\to\) ... residuum \(\to\) of \(\land\)

0 ... 0

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:

\(\neg\) ... \(\neg\), where \(\neg \alpha = \alpha \to 0\)

1 ... 1

\(\leftrightarrow\) ... \(\leftrightarrow\), where \(\alpha \leftrightarrow \beta = (\alpha \to \beta) \land (\beta \to \alpha)\)

\(\land\) ... \(\land\) = min

\(\lor\) ... \(\lor\) = max

**Exercise**
Verify that the interpretation of \(\land\), \(\lor\) is independent of the choice of the fuzzy conjunction.
BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

\( A \) ... countable set of propositional variables
\( \mathcal{L} = \{\to, 0, \wedge\} \) ... the set of logical connectives:
\( \to \) ... (binary) implication
0 ... (nulary) false
\( \wedge \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\( \neg A = A \to 0 \) ... (unary) negation
\( 1 = \neg 0 = 0 \to 0 \) ... (nulary) true
\( A \leftrightarrow B = (A \to B) \wedge (B \to A) \) ... (binary) equivalence
\( A \triangleleft B = A \wedge (A \to B) \)
\( A \triangledown B = ((A \to B) \to B) \wedge ((B \to A) \to A) \)
no \( A \lor B \) in general
BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1
\( \mathcal{A} \) ... countable set of propositional variables
\( \mathcal{L} = \{ \rightarrow, 0, \wedge \} \) ... the set of logical connectives:
\( \rightarrow \) ... (binary) implication
0 ... (nullary) false
\( \wedge \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\( \neg A = A \rightarrow 0 \) ... (unary) negation
1 = \( \neg 0 = 0 \rightarrow 0 \) ... (nullary) true
\( A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) \) ... (binary) equivalence

\( A \wedge_s B = A \wedge (A \rightarrow B) \)
\( A \vee_s B = ((A \rightarrow B) \rightarrow B) \wedge_s ((B \rightarrow A) \rightarrow A) \)
o \( A \vee B \) in general
**SEMANTICS OF BASIC LOGIC**

In general: a BL-algebra, here only

**Standard semantics** the set of truth values ... \([0, 1]\)

\(\land \ldots\) continuous fuzzy conjunction \(\land\)

\(\rightarrow \ldots\) residuum \(\Rightarrow\) of \(\land\)

0 ... 0

Even the standard semantics **is not unique**, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:

\(\neg \ldots\neg\), where \(\neg \alpha = \alpha \rightarrow 0\)

1 ... 1

\(\leftrightarrow \ldots\leftrightarrow\), where \(\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)\)

\(\bigwedge \ldots\bigwedge \equiv \text{min}\)

\(\bigvee \ldots\bigvee \equiv \text{max}\)

**Exercise** Verify that the interpretation of \(\bigwedge \bigvee\) is independent of the choice of the fuzzy conjunction.
SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

**Standard semantics** the set of truth values ... [0, 1]
∧ ... continuous fuzzy conjunction ∧
→ ... residuum → of ∧
0 ... 0

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

**Interpretation of derived connectives:**
¬ ... ¬, where ¬α = α → 0
1 ... 1
↔ ... ↔, where α ↔ β = (α → β) ∧ (β → α)
∧_S ... ∧_S = min
S ... S = max

**Exercise** Verify that the interpretation of ∧_S, S is independent of the choice of the fuzzy conjunction.
An evaluation (truth assignment) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\land$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:  

**1-tautology** is a formula $A$ which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\land$ and its residuum as an interpretation of $\rightarrow$).

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$,  

$\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T} : e(B) = 1$. 
An **evaluation (truth assignment)** can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\land$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:

- **1-tautology** is a formula $A$ which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\land$ and its residuum as an interpretation of $\rightarrow$).

**Notation:** $\models A$

Moreover, for any theory $\mathcal{T}$,

$\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T} : e(B) = 1$.
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)
(A2) \(A \land B \rightarrow A\)
(A3) \(A \land B \rightarrow B \land A\)
(A4) \(A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\)
(A5a) \((A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C')\)
(A5b) \((A \land B \rightarrow C') \rightarrow (A \rightarrow (B \rightarrow C'))\)
(A6) \(((A \rightarrow B) \rightarrow C') \rightarrow (((B \rightarrow A) \rightarrow C') \rightarrow C)\)
(A7) \(0 \rightarrow A\)
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)

(A2) \(A \land B \rightarrow A\)

(A3) \(A \land B \rightarrow B \land A\)

(A4) \(A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\)

(A5a) \((A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C')\)

(A5b) \((A \land B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C'))\)

(A6) \(((A \rightarrow B) \rightarrow C') \rightarrow (((B \rightarrow A) \rightarrow C') \rightarrow C')\)

(A7) \(0 \rightarrow A\)
An **evaluation** (truth assignment) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\land$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:

**1-tautology** is a formula $A$ which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\land$ and its residuum as an interpretation of $\to$)

Notation: $\models A$

Moreover, for any theory $T$, $T \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in T : e(B) = 1$
An **evaluation** (truth assignment) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\land$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:

**1-tautology** is a formula $A$ which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\land$ and its residuum as an interpretation of $\to$)

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$, $\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T} : e(B) = 1$
SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

**Standard semantics**

- the set of truth values ... \([0, 1]\)
- \(\land\) ... continuous fuzzy conjunction \(\land\)
- \(\to\) ... residuum \(\to\) of \(\land\)
- 0 ... 0

Even the standard semantics **is not unique**, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:

- \(\neg\) ... \(\neg\), where \(\neg \alpha = \alpha \to 0\)
- 1 ... 1
- \(\leftrightarrow\) ... \(\leftrightarrow\), where \(\alpha \leftrightarrow \beta = (\alpha \to \beta) \land (\beta \to \alpha)\)

\[\land_{\bar{S}} ... \land_{\bar{S}} = \text{min}\]
\[\lor_{\bar{S}} ... \lor_{\bar{S}} = \text{max}\]

**Exercise**

Verify that the interpretation of \(\land_{\bar{S}}, \lor_{\bar{S}}\) is independent of the choice of the fuzzy conjunction.
SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

Standard semantics

the set of truth values \[0, 1]\n\[\land \ldots \text{continuous fuzzy conjunction } \land\]
\[\rightarrow \ldots \text{residuum } \rightarrow \text{ of } \land\]
\[0 \ldots 0\]

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:
\[\neg \ldots \neg, \quad \text{where } \neg \alpha = \alpha \rightarrow 0\]
\[1 \ldots 1\]
\[\leftrightarrow \ldots \leftrightarrow, \quad \text{where } \alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)\]
\[\land \_ \ldots \land = \min\]
\[\lor \_ \ldots \lor = \max\]

Exercise
Verify that the interpretation of \(\land\_\), \(\lor\_\) is independent of the choice of the fuzzy conjunction.
BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

\( \mathcal{A} \) ... countable set of propositional variables
\( \mathcal{L} = \{ \rightarrow, 0, \wedge \} \) ... the set of logical connectives:
\( \rightarrow \) ... (binary) implication
\( 0 \) ... (nullary) false
\( \wedge \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\( \neg A = A \rightarrow 0 \) ... (unary) negation
\( 1 = \neg 0 = 0 \rightarrow 0 \) ... (nullary) true
\( A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) \) ... (binary) equivalence
\( \begin{align*}
A \wedge S B &= A \wedge (A \rightarrow B) \\
A \wedge S B &= ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)
\end{align*} 

no \( A \vee B \) in general
BASIC LOGIC (BL) 
as an example of a many-valued propositional logic

SYNTAX OF BASIC LOGIC 1

\( \mathcal{A} \) ... countable set of propositional variables
\[ \mathcal{L} = \{ \to, 0, \land \} \] ... the set of logical connectives:
\( \to \) ... (binary) implication
0 ... (nullary) false
\( \land \) ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
\( \neg A = A \to 0 \) ... (unary) negation
1 = \( \neg 0 = 0 \to 0 \) ... (nullary) true
\( A \leftrightarrow B = (A \to B) \land (B \to A) \) ... (binary) equivalence
\( A \triangleleft B = A \land (A \to B) \)
\( A \triangledown B = ((A \to B) \to B) \land ((B \to A) \to A) \)

No \( A \triangledown B \) in general
SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

Standard semantics the set of truth values ... [0, 1]
\( \wedge \) ... continuous fuzzy conjunction \( \wedge \)
\( \rightarrow \) ... residuum \( \Rightarrow \) of \( \wedge \)
0 ... 0

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:
\( \neg \) ... \( \neg \), where \( \neg \alpha = \alpha \rightarrow 0 \)
1 ... 1
\( \leftrightarrow \) ... \( \leftrightarrow \), where \( \alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha) \)
\( \wedge^{S} \) ... \( \wedge^{S} = \text{min} \)
\( \wedge^{S} \) ...
\( \vee^{S} \) ... \( \vee^{S} = \text{max} \)

Exercise Verify that the interpretation of \( \wedge^{S}, \vee^{S} \) is independent of the choice of the fuzzy conjunction.
SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

**Standard semantics**
the set of truth values ... $[0, 1]$
$\wedge$ ... continuous fuzzy conjunction $\wedge$
$\rightarrow$ ... residuum $\Rightarrow$ of $\wedge$
$0$ ... $0$

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:
$\neg$ ... $\neg$, where $\neg \alpha = \alpha \rightarrow 0$
$1$ ... $1$
$\leftrightarrow$ ... $\leftrightarrow$, where $\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$
$\wedge_{S}$ ... $\wedge_{S} = \text{min}$
$\vee_{S}$ ... $\vee_{S} = \text{max}$

**Exercise**
Verify that the interpretation of $\wedge_{S}, \vee_{S}$ is independent of the choice of the fuzzy conjunction.
An **evaluation (truth assignment)** can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\land$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:

1-tautology is a formula $A$ which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\land$ and its residuum as an interpretation of $\to$)

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$,

$\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T}: e(B) = 1$
An **evaluation (truth assignment)** can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\land$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:

**1-tautology** is a formula $A$ which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\land$ and its residuum as an interpretation of $\rightarrow$)

Notation: $\models A$

Moreover, for any theory $T$,

$T \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in T : e(B) = 1$.

\[
\begin{align*}
0.2 & \rightarrow 1 \\
0.5 & \rightarrow (b \land c) \\
0.5 & \rightarrow 1 \\
1 & \rightarrow (b \land c)
\end{align*}
\]
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \[(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\]

(A2) \[A \land B \rightarrow A\]

(A3) \[A \land B \rightarrow B \land A\]

(A4) \[A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\]

(A5a) \[(A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C)\]

(A5b) \[(A \land B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))\]

(A6) \[((A \rightarrow B) \rightarrow C) \rightarrow (((B \rightarrow A) \rightarrow C) \rightarrow C)\]

(A7) \[0 \rightarrow A\]
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)

(A2) \(A \land B \rightarrow A\)

(A3) \(A \land B \rightarrow B \land A\)

(A4) \(A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\)

(A5a) \((A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C')\)

(A5b) \((A \land B \rightarrow C') \rightarrow (A \rightarrow (B \rightarrow C'))\)

(A6) \(((A \rightarrow B) \rightarrow C') \rightarrow (((B \rightarrow A) \rightarrow C') \rightarrow C')\)

(A7) \(0 \rightarrow A\)
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)

(A2) \(A \land B \rightarrow A\)

(A3) \(A \land B \rightarrow B \land A\)

(A4) \(A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\)

(A5a) \((A \rightarrow (B \rightarrow C')) \rightarrow (A \land B \rightarrow C')\)

(A5b) \((A \land B \rightarrow C') \rightarrow (A \rightarrow (B \rightarrow C'))\)

(A6) \(((A \rightarrow B) \rightarrow C') \rightarrow (((B \rightarrow A) \rightarrow C') \rightarrow C')\)

(A7) \(0 \rightarrow A\)
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)

(A2) \(A \land B \rightarrow A\)

(A3) \(A \land B \rightarrow B \land A\)

(A4) \(A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\)

(A5a) \((A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C')\)

(A5b) \((A \land B \rightarrow C') \rightarrow (A \rightarrow (B \rightarrow C'))\)

(A6) \(((A \rightarrow B) \rightarrow C') \rightarrow (((B \rightarrow A) \rightarrow C') \rightarrow C')\)

(A7) \(0 \rightarrow A\)

Deduction rule: Modus Ponens \(\text{MP}(A, A \rightarrow B):\)
\[
\frac{A, A \rightarrow B}{B}
\]

Theory = set of formulas (special axioms)

Proofs and provable formulas (=theorems) are defined as usual
Notation: ⊢ A,  \mathcal{T} \vdash A

Example 1  
(C1)  \ A \rightarrow (B \rightarrow A) \quad \text{is provable in BL:}

\begin{align*}
(A2): & \quad D_1 = A \land B \rightarrow A \\
(A5b), C := A: & \quad D_2 = (A \land B \rightarrow A) \rightarrow (A \rightarrow (B \rightarrow A)) \\
\text{MP}(D_1, D_2): & \quad D_3 = A \rightarrow (B \rightarrow A)
\end{align*}

\Rightarrow (C1) \text{ can be added to axioms of BL}

Proposition 1  
Consequence of (A1):

\{ A \rightarrow B, B \rightarrow C \} \vdash A \rightarrow C

\Rightarrow \text{we can add a deduction rule}

\text{Tl}(A \rightarrow B, B \rightarrow C): \quad \frac{A \rightarrow B, B \rightarrow C}{A \rightarrow C} \quad \text{(transitivity of implication)}
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)
(A2) \(A \land B \rightarrow A\)
(A3) \(A \land B \rightarrow B \land A\)
(A4) \(A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\)
(A5a) \((A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C')\)
(A5b) \((A \land B \rightarrow C') \rightarrow (A \rightarrow (B \rightarrow C'))\)
(A6) \(((A \rightarrow B) \rightarrow C') \rightarrow (((B \rightarrow A) \rightarrow C') \rightarrow C')\)
(A7) \(0 \rightarrow A\)

Deduction rule: Modus Ponens \(\text{MP}(A, A \rightarrow B) : \begin{array}{c} A, A \rightarrow B \hline B \end{array}\)

Theory = set of formulas (special axioms)

Proofs and provable formulas (=theorems) are defined as usual
Notation: \( \vdash A, \quad \mathcal{T} \vdash A \)

**Example 1**  
(C1) \( A \rightarrow (B \rightarrow A) \) is provable in BL:

(A2): \( D_1 = A \land B \rightarrow A \)

(A5b), \( C := A \): \( D_2 = (A \land B \rightarrow A) \rightarrow (A \rightarrow (B \rightarrow A)) \)

\( \text{MP}(D_1, D_2) \): \( D_3 = A \rightarrow (B \rightarrow A) \)

\( \Rightarrow \) (C1) can be added to axioms of BL

**Proposition 1**  
Consequence of (A1):

\( \{A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C \)

\( \Rightarrow \) we can add a deduction rule

\( \text{TI}(A \rightarrow B, B \rightarrow C): \frac{A \rightarrow B, B \rightarrow C}{A \rightarrow C} \)  \( (\text{transitivity of implication}) \)
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)

(A2) \(A \land B \rightarrow A\)

(A3) \(A \land B \rightarrow B \land A\)

(A4) \(A \land (A \rightarrow B) \rightarrow B \land (B \rightarrow A)\)

(A5a) \((A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C')\)

(A5b) \((A \land B \rightarrow C') \rightarrow (A \rightarrow (B \rightarrow C'))\)

(A6) \(((A \rightarrow B) \rightarrow C') \rightarrow (((B \rightarrow A) \rightarrow C') \rightarrow C')\)

(A7) \(0 \rightarrow A\)

Deduction rule: Modus Ponens

\[\text{MP}(A, A \rightarrow B) : \quad \frac{A, \quad A \rightarrow B}{B} \]

Theory = set of formulas (special axioms)

Proofs and provable formulas (=theorems) are defined as usual
SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1) \((A \to B) \to ((B \to C) \to (A \to C))\)
(A2) \(A \land B \to A\)
(A3) \(A \land B \to B \land A\)
(A4) \(A \land (A \to B) \to B \land (B \to A)\)
(A5a) \((A \to (B \to C)) \to (A \land B \to C')\)
(A5b) \((A \land B \to C') \to (A \to (B \to C'))\)
(A6) \(((A \to B) \to C') \to (((B \to A) \to C) \to C)\)
(A7) \(0 \to A\)

Deduction rule: Modus Ponens

\[
\text{MP}(A, A \to B) : \quad \frac{A, A \to B}{B}
\]

Theory = set of formulas (special axioms)

Proofs and provable formulas (=theorems) are defined as usual
Notation: \( \vdash A, \quad \mathcal{T} \vdash A \)

**Example 1**  
(C1) \( A \rightarrow (B \rightarrow A) \) is provable in BL:

\[
\begin{align*}
(A2) : & \quad D_1 = A \land B \rightarrow A \\
(A5b), C := A : & \quad D_2 = (A \land B \rightarrow A) \rightarrow (A \rightarrow (B \rightarrow A)) \\
\text{MP}(D_1, D_2) : & \quad D_3 = A \rightarrow (B \rightarrow A)
\end{align*}
\]

\( \Rightarrow \) (C1) can be added to axioms of BL

**Proposition 1**  
Consequence of (A1):

\[ \{ A \rightarrow B, \ B \rightarrow C \} \vdash A \rightarrow C \]

\( \Rightarrow \) we can add a deduction rule

TI\((A \rightarrow B, \ B \rightarrow C)\): \[ \frac{A \rightarrow B, \ B \rightarrow C}{A \rightarrow C} \] (transitivity of implication)
Notation: \( \vdash A, \quad \mathcal{T} \vdash A \)

**Example 1**  
(C1) \( A \rightarrow (B \rightarrow A) \) is provable in BL:

1. (A2) \( D_1 = A \land B \rightarrow A \)
2. (A5b) \( C := A \): \( D_2 = (A \land B \rightarrow A) \rightarrow (A \rightarrow (B \rightarrow A)) \)
3. MP\((D_1, D_2)\): \( D_3 = A \rightarrow (B \rightarrow A) \)

\( \Rightarrow \) (C1) can be added to axioms of BL

**Proposition 1**  
Consequence of (A1):

\[ \{A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C \]

\( \Rightarrow \) we can add a deduction rule

\( \text{TI}(A \rightarrow B, B \rightarrow C): \frac{A \rightarrow B, B \rightarrow C}{A \rightarrow C} \) (transitivity of implication)

(A1): \( (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \)

MP: \( (B \rightarrow C) \rightarrow (A \rightarrow C) \)

MP: \( A \rightarrow C \)
Example 2  ⊢ (A → (B → C)) → (B → (A → C))
(Exchange rule, also called “exchange axiom”)

(A1) A := B ∧ A,
    B := A ∧ B :  D₁ = (B ∧ A → A ∧ B) → ((A ∧ B → C) → (B ∧ A → C))
(A3) A :=: B :  D₂ = B ∧ A → A ∧ B
MP(D₂, D₃) :  D₃ = (A ∧ B → C) → (B ∧ A → C)
    (A5a) :  D₄ = (A → (B → C)) → (A ∧ B → C)
(A5b) A :=: B :  D₅ = (B ∧ A → C) → (B → (A → C))
TI(D₄, D₃) :  D₆ = (A → (B → C)) → (B ∧ A → C)
TI(D₆, D₅) :  D₇ = (A → (B → C)) → (B → (A → C))
Example 3 \[ \vdash A \rightarrow A \]

For brevity, let \( B \) denote a provable formula, e.g., axiom (A1).

(A1) \[ D_1 = B \]

Ex. 2, \( C := A \) : \[ D_2 = (A \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow (A \rightarrow A)) \]

(C1) \[ D_3 = A \rightarrow (B \rightarrow A) \]

MP(\( D_3, D_2 \)) \[ D_4 = B \rightarrow (A \rightarrow A) \]

MP(\( D_1, D_4 \)) \[ D_5 = A \rightarrow A \]
Example 2  \( \vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \)
(Exchange rule, also called “exchange axiom”)

(A1) \( A := B \land A \),

\( B := A \land B : \quad D_1 = (B \land A \rightarrow A \land B) \rightarrow ((A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C)) \)

(A3) \( A ::= B : \quad D_2 = B \land A \rightarrow A \land B \)

MP(\( D_2 , D_3 \)) :

\( D_3 = (A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C) \)

(A5a) :

\( D_4 = (A \rightarrow (B \rightarrow C')) \rightarrow (A \land B \rightarrow C') \)

(A5b) \( A ::= B : \quad D_5 = (B \land A \rightarrow C') \rightarrow (B \rightarrow (A \rightarrow C)) \)

TI(\( D_4 , D_3 \)) :

\( D_6 = (A \rightarrow (B \rightarrow C')) \rightarrow (B \land A \rightarrow C) \)

TI(\( D_6 , D_5 \)) :

\( D_7 = (A \rightarrow (B \rightarrow C')) \rightarrow (B \rightarrow (A \rightarrow C)) \)
Example 2 \( \vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \)
(Exchange rule, also called “exchange axiom”)

(A1) \( A := B \land A \)  
\( B := A \land B \)  
\( D_1 = (B \land A \rightarrow A \land B) \rightarrow ((A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C)) \)

(A3) \( A := B \)  
\( B := A \land B \)  
\( D_2 = B \land A \rightarrow A \land B \)

MP(\( D_2, D_3 \))  
\( D_3 = (A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C) \)

(A5a)  
\( D_4 = (A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C) \)

(A5b) \( A := B \)  
\( B := A \land B \)  
\( D_5 = (B \land A \rightarrow C) \rightarrow (B \rightarrow (A \rightarrow C)) \)

TI(\( D_4, D_3 \))  
\( D_6 = (A \rightarrow (B \rightarrow C)) \rightarrow (B \land A \rightarrow C) \)

TI(\( D_6, D_5 \))  
\( D_7 = (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \)
Example 3 \[ \vdash A \rightarrow A \]

For brevity, let \( B \) denote a provable formula, e.g., axiom (A1).

(A1) : \[ D_1 = B \]

Ex. 2, \( C := A \) : \[ D_2 = (A \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow (A \rightarrow A)) \]

(C1) : \[ D_3 = A \rightarrow (B \rightarrow A) \]

MP(\( D_3, D_2 \)) : \[ D_4 = B \rightarrow (A \rightarrow A) \]

MP(\( D_1, D_4 \)) : \[ D_5 = A \rightarrow A \]
Example 3 \[ \vdash A \rightarrow A \]

For brevity, let $B$ denote a provable formula, e.g., axiom (A1).

(A1) : \[ D_1 = B \]

Ex. 2, $C := A$ : \[ D_2 = (A \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow (A \rightarrow A)) \]

(C1) : \[ D_3 = A \rightarrow (B \rightarrow A) \]

MP($D_3$, $D_2$) : \[ D_4 = B \rightarrow (A \rightarrow A) \]

MP($D_1$, $D_4$) : \[ D_5 = A \rightarrow A \]
Example 2

\( \vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \)

(Exchange rule, also called “exchange axiom”)

(A1) \( A := B \land A \)

\( B := A \land B \) : \( D_1 = (B \land A \rightarrow A \land B) \rightarrow ((A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C)) \)

(A3) \( A ::= B \) : \( D_2 = B \land A \rightarrow A \land B \)

\( \text{MP}(D_2, D_3) : D_3 = (A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C) \)

(A5a) : \( D_4 = (A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C) \)

(A5b) \( A ::= B \) : \( D_5 = (B \land A \rightarrow C) \rightarrow (B \rightarrow (A \rightarrow C)) \)

\( \text{TI}(D_4, D_3) : D_6 = (A \rightarrow (B \rightarrow C)) \rightarrow (B \land A \rightarrow C) \)

\( \text{TI}(D_6, D_5) : D_7 = (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \)
Example 2
\[ \vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \]
(Exchange rule, also called “exchange axiom”)

(A1) \( A := B \land A \)

\( B := A \land B \)

\[ D_1 = (B \land A \rightarrow A \land B) \rightarrow ((A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C)) \]

(A3) \( A ::= B \)

\[ D_2 = B \land A \rightarrow A \land B \]

MP(\( D_2, D_3 \)):

\[ D_3 = (A \land B \rightarrow C) \rightarrow (B \land A \rightarrow C) \]

(A5a):

\[ D_4 = (A \rightarrow (B \rightarrow C)) \rightarrow (A \land B \rightarrow C) \]

(A5b) \( A ::= B \)

\[ D_5 = (B \land A \rightarrow C) \rightarrow (B \rightarrow (A \rightarrow C)) \]

TI(\( D_4, D_3 \)):

\[ D_6 = (A \rightarrow (B \rightarrow C)) \rightarrow (B \land A \rightarrow C) \]

TI(\( D_6, D_5 \)):

\[ D_7 = (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \]

\[ 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \]
Example 3 \[ \vdash A \rightarrow A \]

For brevity, let $B$ denote a provable formula, e.g., axiom (A1).

(A1): \[ D_1 = B \]

Ex. 2, $C := A$:\[ D_2 = (A \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow (A \rightarrow A)) \]

(C1): \[ D_3 = A \rightarrow (B \rightarrow A) \]

MP($D_3, D_2$): \[ D_4 = B \rightarrow (A \rightarrow A) \]

MP($D_1, D_4$): \[ D_5 = A \rightarrow A \]
Example 3 \[ \vdash A \rightarrow A \]

For brevity, let \( B \) denote a provable formula, e.g., axiom (A1).

(A1): \[ D_1 = B \]

Ex. 2, \( C := A \): \[ D_2 = (A \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow (A \rightarrow A)) \]

(C1): \[ D_3 = A \rightarrow (B \rightarrow A) \]

MP\((D_3, D_2)\): \[ D_4 = B \rightarrow (A \rightarrow A) \]

MP\((D_1, D_4)\): \[ D_5 = A \rightarrow A \]