Deduction theorem in classical logic

$\mathcal{T}$ ... theory

A, B ... formulas

$$\boxed{\mathcal{T} \cup \{A\} \vdash B \text{ iff } \mathcal{T} \vdash A \to B}$$

Proof $\Leftarrow$:

	//BEGIN of proof of $\mathcal{T} \vdash A \to B$
	$\vdots$
	$D_{i-1} = A \to B$
	//END of proof of $\mathcal{T} \vdash A \to B$
SA :	$D_i = A$
MP(D_i, D_{i-1}) :	$D_{i+1} = B$

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Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \to B$.

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Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \rightarrow B$.

1. B is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

$$\begin{array}{rl} & D_1 = B \\ RI(D_1): & D_2 = A \rightarrow B \end{array}$$

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2. $B \neq A$ because $\mathcal{T} \vdash A \to A$.

3. B is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.
WLOG, we choose for B a formula with the shortest possible proof; its shortest proof must be of the following form:

$$\begin{array}{rl} & \vdots \\ & D_i \\ & \vdots \\ & D_j = D_i \to B \\ & \vdots \\ \mathsf{MP}(D_i, D_j): & D_m = B \end{array}$$

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The proofs of $\mathcal{T} \cup \{A\} \vdash D_i$, $\mathcal{T} \cup \{A\} \vdash D_j$ are of lengths $< m$, therefore

$$\mathcal{T} \vdash A \to D_i$$
$$\mathcal{T} \vdash A \to D_j = A \to (D_i \to B)$$

Proof of $\mathcal{T} \vdash A \to B$:

$$
\begin{array}{rl}
 & //\text{BEGIN of proof of } \mathcal{T} \vdash A \to D_i \\
 & \vdots \\
 & D_k = A \to D_i \\
 & //\text{END of proof of } \mathcal{T} \vdash A \to D_i \\
 & //\text{BEGIN of proof of } \mathcal{T} \vdash A \to D_j \\
 & \vdots \\
 & D_n = A \to \overbrace{(D_i \to B)}^{D_j} \\
 & //\text{END of proof of } \mathcal{T} \vdash A \to D_j \\
\text{(C2) } B := D_i, C := B: & D_{n+1} = (A \to (D_i \to B)) \to ((A \to D_i) \to (A \to B)) \\
\text{MP}(D_n, D_{n+1}): & D_{n+2} = (A \to D_i) \to (A \to B) \\
\text{MP}(D_k, D_{n+2}): & D_{n+3} = A \to B
\end{array}
$$

Corollary Cor2 $A \vdash A \vee B$ for all A, B

$$A \vdash \neg A \rightarrow B = A \vee B$$

$$\Updownarrow \text{(DT)}$$

$$\text{ALL}(A): \quad \{A, \neg A\} \vdash B$$

$\Rightarrow$ we can add a deduction rule $\dfrac{A}{A \vee B}$ (and $\dfrac{B}{A \vee B}$ was already proved in Cl9)

Corollary Cor3 $\quad A \vdash \neg\neg A, \qquad \vdash A \to \neg\neg A$ for all A

$$\vdash A \to \overbrace{(\neg A \to \mathbf{0})}^{\neg\neg A}$$
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Corollary Cor4 $\quad \neg\neg A \vdash A, \qquad \vdash \neg\neg A \to A$ for all A

$$\begin{aligned}
\text{Cor3, } A := \neg A: &\quad D_1 = \neg A \to \neg\neg\neg A \\
\text{(C3) } B := \neg\neg A: &\quad D_2 := (\neg A \to \neg\neg\neg A) \to (\neg\neg A \to A) \\
\text{MP}(D_1, D_2): &\quad D_3 = \neg\neg A \to A
\end{aligned}$$

Corollary Cor5 $\quad \vdash A \leftrightarrow \neg\neg A \qquad$ (can be added to axioms)

How can we simplify our proofs?

$B \leftrightarrow C \vdash (A \to B) \leftrightarrow (A \to C)$

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INTERPLAY OF SYNTAX AND SEMANTICS OF CLASSICAL LOGIC

Weak soundness Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.

Strong soundness For any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness Each tautology is provable, i.e., if $\models A$, then $\vdash A$.

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AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

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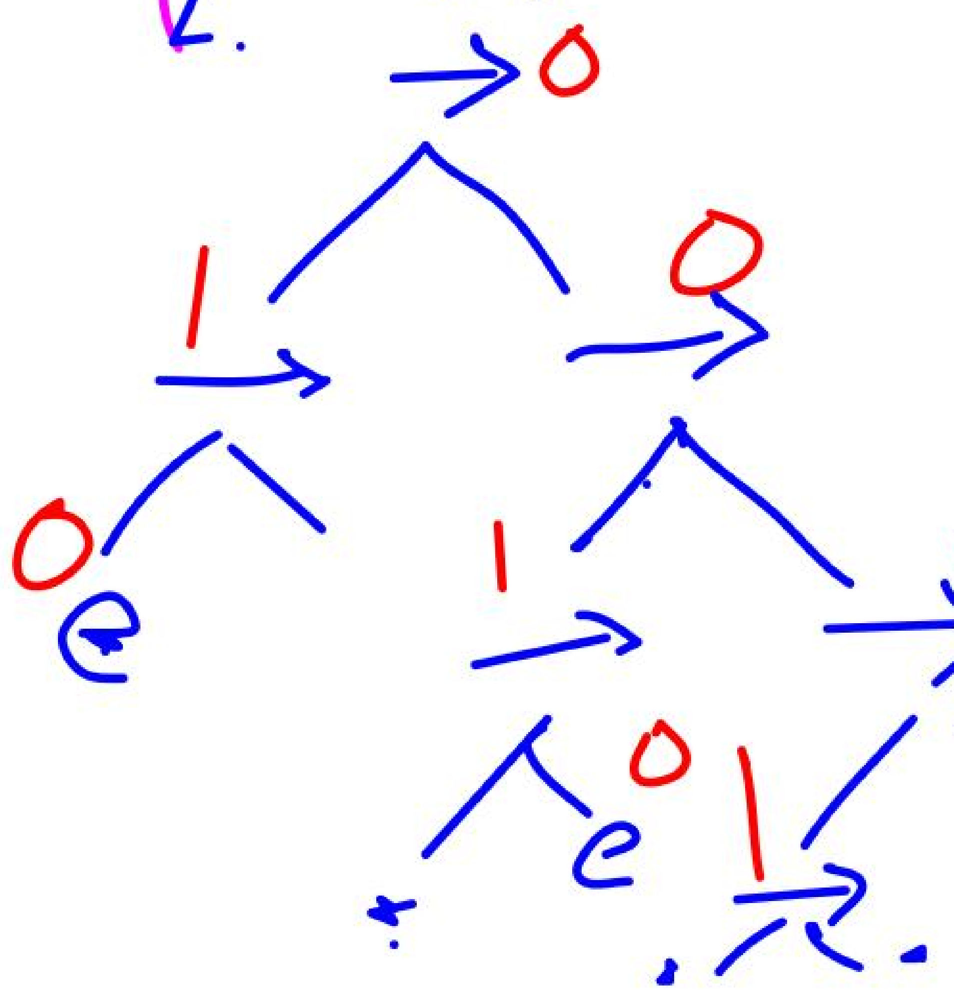
Is A a tautology?

A		e(A)
0	0	1
0	1	1
1	0	1
1	1	1

2.

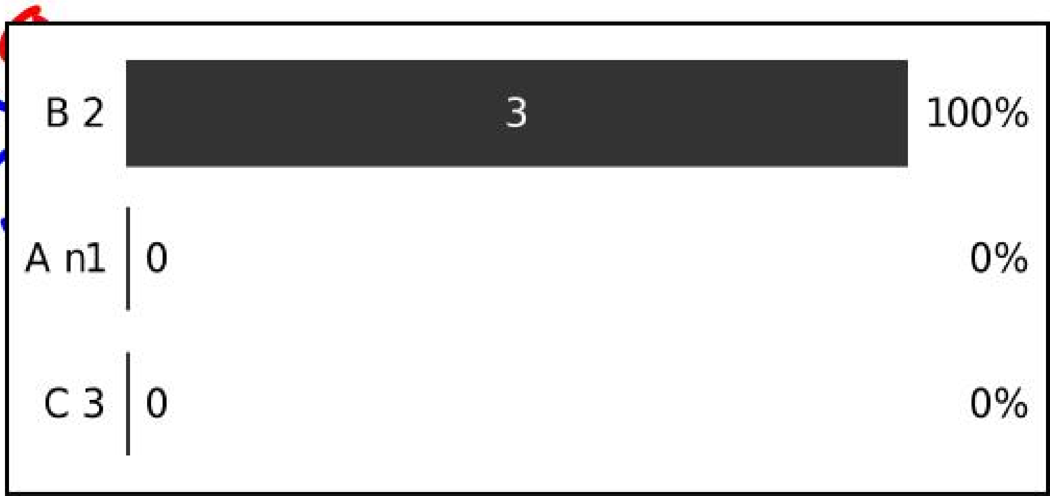


C3. axiom...

⊣ - ...

deduce: ...

...



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Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

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An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\wedge$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:
1-tautology is a formula A which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\wedge$ and its residuum as an interpretation of $\rightarrow$)

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$,
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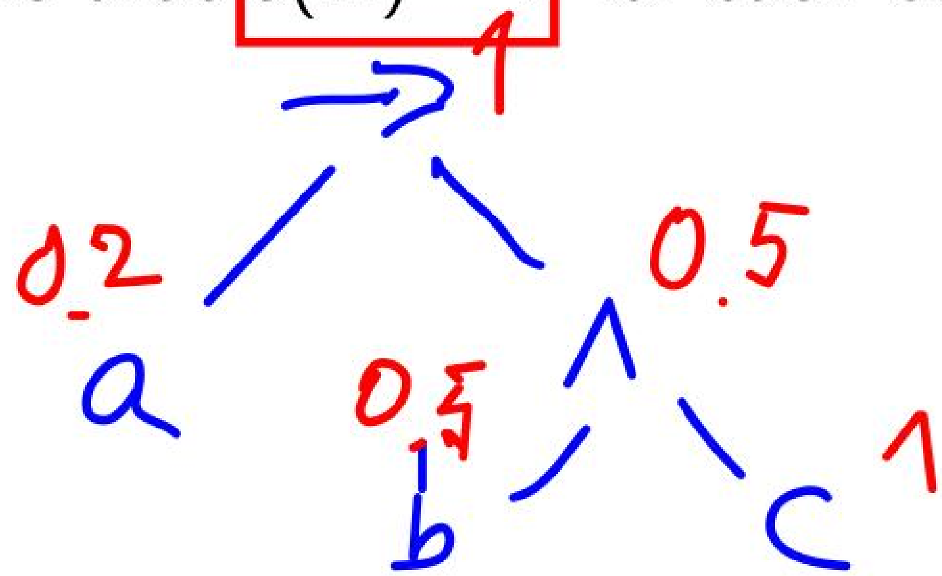

$a \rightarrow (b \wedge c)$

SYNTAX OF BASIC LOGIC 2

Logical axioms

(A1)	$(A \to B) \to ((B \to C) \to (A \to C))$
(A2)	$A \wedge B \to A$
(A3)	$A \wedge B \to B \wedge A$
(A4)	$A \wedge (A \to B) \to B \wedge (B \to A)$
(A5a)	$(A \to (B \to C)) \to (A \wedge B \to C)$
(A5b)	$(A \wedge B \to C) \to (A \to (B \to C))$
(A6)	$((A \to B) \to C) \to (((B \to A) \to C) \to C)$
(A7)	$\mathbf{0} \to A$

SYNTAX OF BASIC LOGIC 2

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left] & B \arrow[r] & C \end{tikzcd}$$

Logical axioms

(A1)	$(A \to B) \to ((B \to C) \to (A \to C))$
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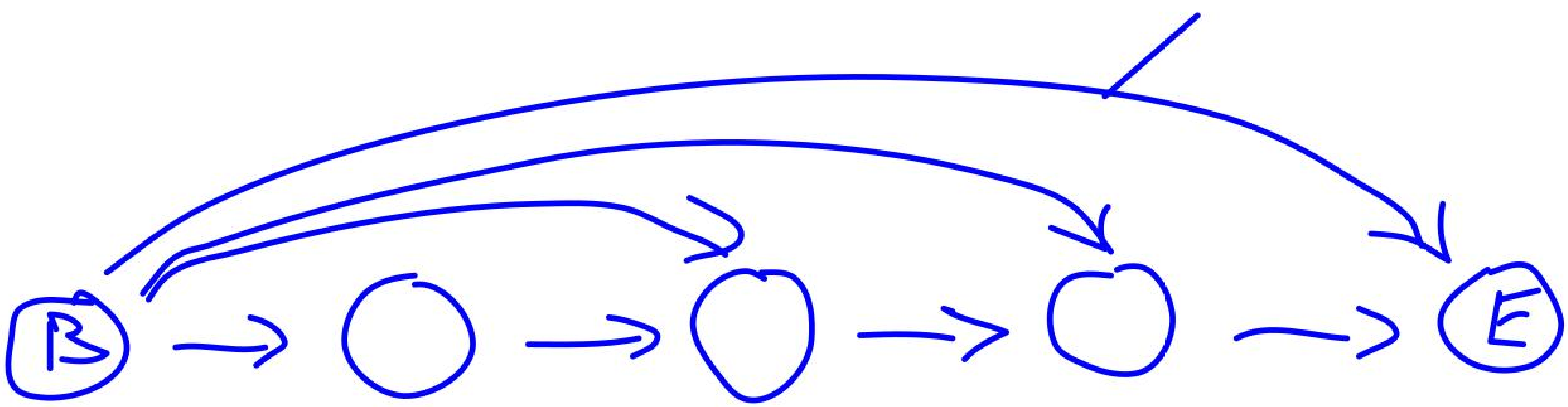
(A7)	$\mathbf{0} \to A$

$(B \wedge A) \to (A \wedge B)$

$A \wedge B \to B \wedge A$



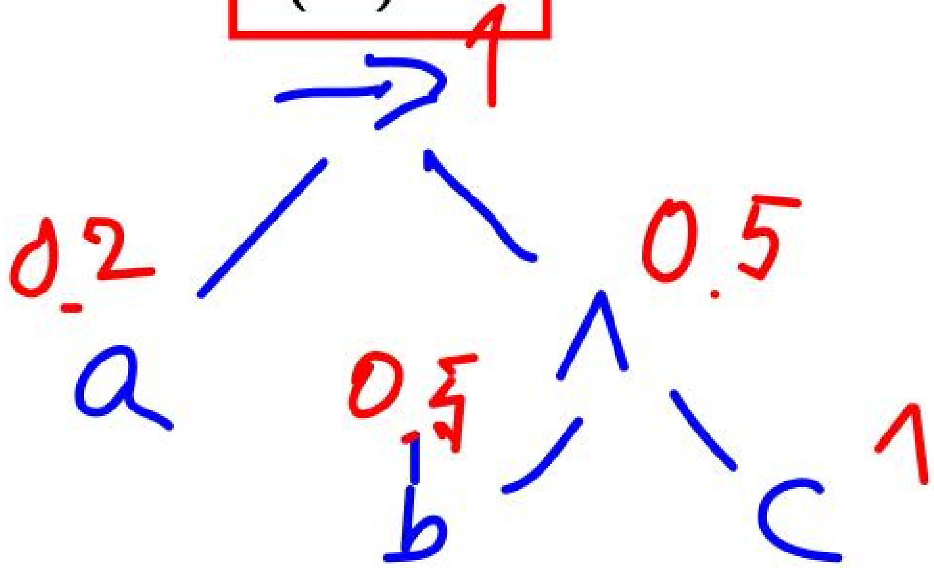
An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\wedge$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:
1-tautology is a formula A which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\wedge$ and its residuum as an interpretation of $\to$)

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$,
$\mathcal{T} \models A$ means that $\boxed{e(A) = 1}$ for each evaluation such that $\boxed{\forall B \in \mathcal{T} : e(B) = 1}$.



$a \to (b \wedge c)$

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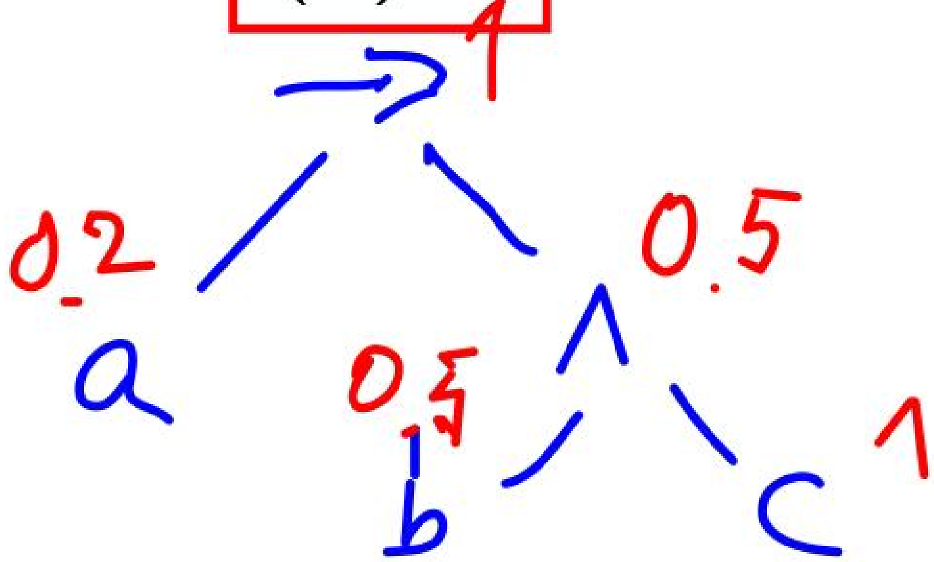

SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

Standard semantics the set of truth values ... $[0,1]$

$\wedge$ ... continuous fuzzy conjunction $\underset{\cdot}{\wedge}$

$\to$ ... residuum $\overset{R}{\underset{\cdot}{\to}}$ of $\underset{\cdot}{\wedge}$

$\mathbf{0}$ ... 0

Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.

Interpretation of derived connectives:

$\neg$ ... $\underset{\cdot}{\neg}$, where $\underset{\cdot}{\neg}\,\alpha = \alpha \underset{\cdot}{\to} 0$

$\mathbf{1}$ ... 1

$\leftrightarrow$ ... $\underset{\cdot}{\leftrightarrow}$, where $\alpha \underset{\cdot}{\leftrightarrow} \beta = (\alpha \underset{\cdot}{\to} \beta) \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\to} \alpha)$

$\underset{S}{\wedge}$ ... $\underset{S}{\wedge} = \min$

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Exercise Verify that the interpretation of $\underset{S}{\wedge}, \overset{S}{\vee}$ is independent of the choice of the fuzzy conjunction.

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BASIC LOGIC (BL)
AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

$\mathcal{A}$ ... countable set of propositional variables

$\mathcal{L} = \{\to, \mathbf{0}, \wedge\}$ ... the set of logical connectives:

$\to$ ... (binary) implication

$\mathbf{0}$ ... (nulary) false

$\wedge$ ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:

$\neg A = A \to \mathbf{0}$ ... (unary) negation

$\mathbf{1} = \neg\mathbf{0} = \mathbf{0} \to \mathbf{0}$ ... (nulary) true

$A \leftrightarrow B = (A \to B) \wedge (B \to A)$ ... (binary) equivalence

$$\boxed{\begin{array}{l} A \underset{S}{\wedge} B = A \wedge (A \to B) \\ A \overset{S}{\vee} B = ((A \to B) \to B) \underset{S}{\wedge} ((B \to A) \to A) \end{array}}$$

no $A \vee B$ in general

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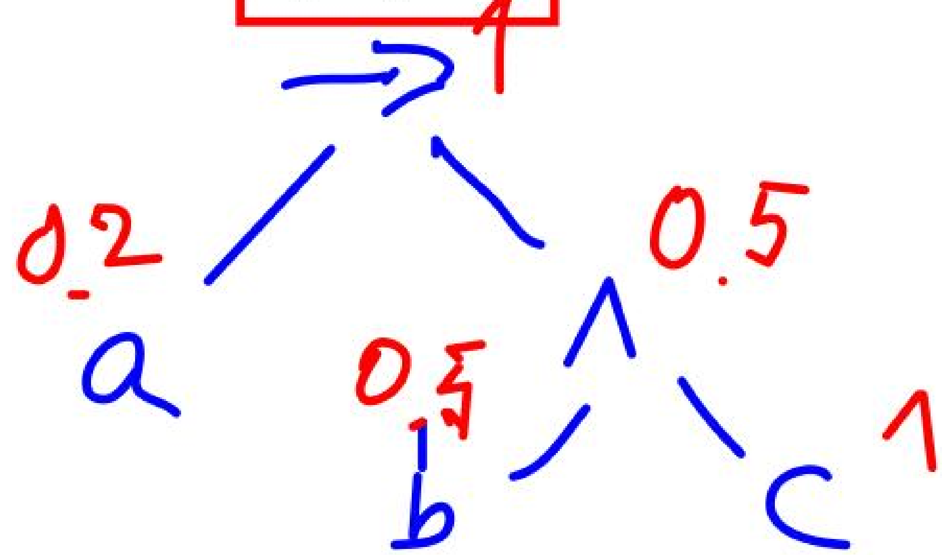

$a \rightarrow (b \wedge c)$

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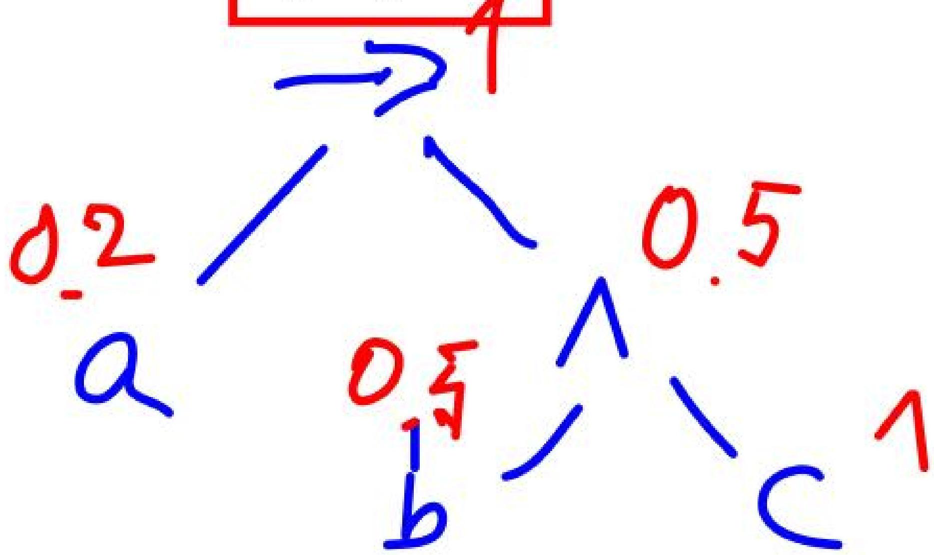

$a \rightarrow (b \wedge c)$

SYNTAX OF BASIC LOGIC 2

\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left] & B \arrow[r] & C
\end{tikzcd}

Logical axioms

(A1)	$(A \to B) \to ((B \to C) \to (A \to C))$
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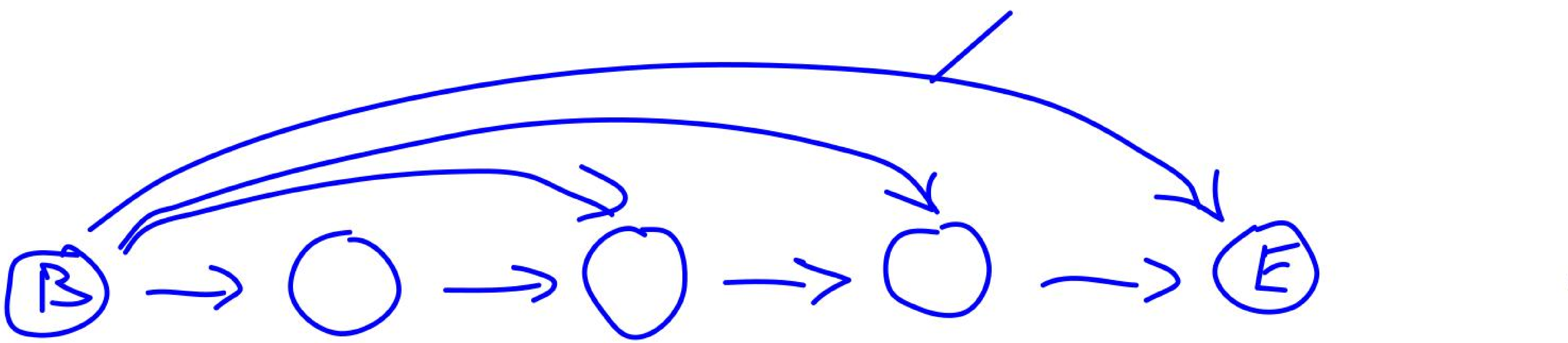

SYNTAX OF BASIC LOGIC 2

$A \to B \to C$

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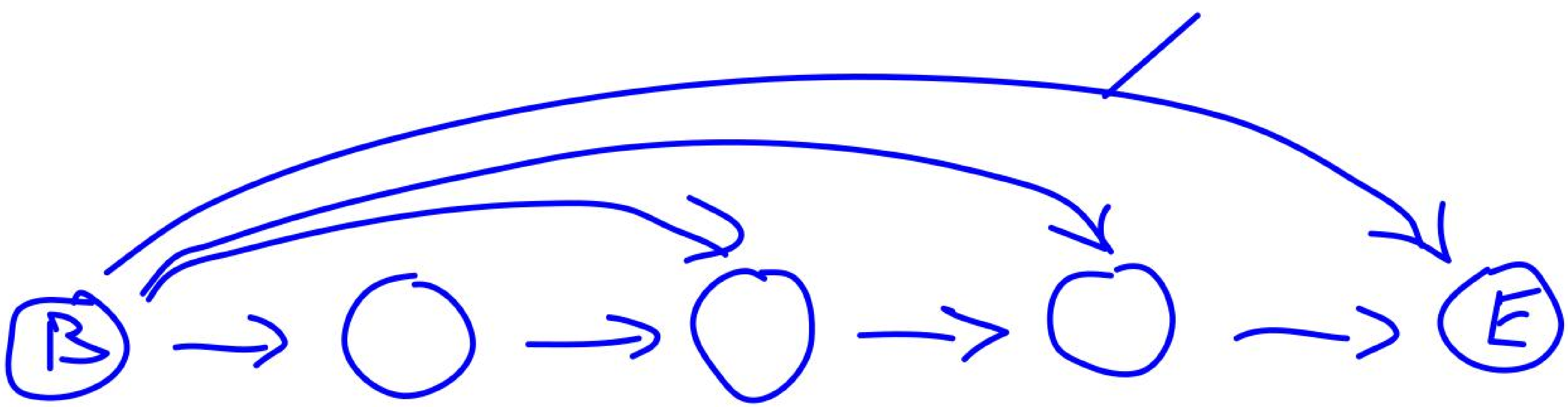

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SYNTAX OF BASIC LOGIC 2

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Deduction rule: Modus Ponens MP(A, $A \to B$) : $\dfrac{A,\ A \to B}{B}$

Theory $=$ set of formulas (**special axioms**)

Proofs and **provable formulas** (=**theorems**) are defined as usual

Notation: $\vdash A, \qquad \mathcal{T} \vdash A$

Example 1 (C1) $A \to (B \to A)$ is provable in BL:

$$\begin{aligned} \text{(A2)}: &\quad D_1 = A \wedge B \to A \\ \text{(A5b)}, C := A: &\quad D_2 = (A \wedge B \to A) \to (A \to (B \to A)) \\ \text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A) \end{aligned}$$

$\Rightarrow$ (C1) can be added to axioms of BL

Proposition 1 Consequence of (A1):

$$\{A \to B,\ B \to C\} \vdash A \to C$$

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TI($A \to B,\ B \to C$): $\dfrac{A \to B,\ B \to C}{A \to C}$ (transitivity of implication)

SYNTAX OF BASIC LOGIC 2

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MP: $(\boxed{B \to C}) \to (A \to C)$

MP: $A \to C$

Example 2 $\quad \vdash (A \to (B \to C)) \to (B \to (A \to C))$

(**Exchange rule**, also called "exchange axiom")

$$
\begin{aligned}
&\text{(A1) } A := B \wedge A, \\
&B := A \wedge B: && D_1 = (B \wedge A \to A \wedge B) \to ((A \wedge B \to C) \to (B \wedge A \to C)) \\
&\text{(A3) } A :=: B: && D_2 = B \wedge A \to A \wedge B \\
&\mathsf{MP}(D_2, D_3): && D_3 = (A \wedge B \to C) \to (B \wedge A \to C) \\
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\end{aligned}
$$

Example 3 $\quad \vdash A \to A$

For brevity, let B denote a provable formula, e.g., axiom (A1).

$$\begin{aligned}
\text{(A1)}: &\quad D_1 = B \\
\text{Ex. 2}, C := A: &\quad D_2 = (A \to (B \to A)) \to (B \to (A \to A)) \\
\text{(C1)}: &\quad D_3 = A \to (B \to A) \\
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\end{aligned}$$

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(A3) $A :=: B$: $\quad D_2 = B \wedge A \to A \wedge B$

$\mathsf{MP}(D_2, D_3)$: $\quad D_3 =$ 2 $(A \wedge B \to C) \to (B \wedge A \to C)$ 3

(A5a) : $\quad D_4 =$ 1 $(A \to (B \to C)) \to (A \wedge B \to C)$ 2

(A5b) $A :=: B$: $\quad D_5 =$ 3 $(B \wedge A \to C) \to (B \to (A \to C))$ 4

$\mathsf{TI}(D_4, D_3)$: $\quad D_6 = (A \to (B \to C)) \to (B \wedge A \to C)$

$\mathsf{TI}(D_6, D_5)$: $\quad D_7 = (A \to (B \to C)) \to (B \to (A \to C))$

1 → 2 → 3 → 4

Example 3 $\quad \vdash A \to A$

For brevity, let B denote a provable formula, e.g., axiom (A1).

$$\begin{aligned}
\text{(A1)}: &\quad D_1 = B \\
\text{Ex. 2}, C := A: &\quad D_2 = (A \to (B \to A)) \to (B \to (A \to A)) \\
\text{(C1)}: &\quad D_3 = A \to (B \to A) \\
\text{MP}(D_3, D_2): &\quad D_4 = B \to (A \to A) \\
\text{MP}(D_1, D_4): &\quad D_5 = A \to A
\end{aligned}$$

Example 3 $\quad \vdash A \to A$

For brevity, let B denote a provable formula, e.g., axiom (A1).

$$\begin{aligned}
\text{(A1)}: &\quad D_1 = B \\
\text{Ex. 2}, C := A: &\quad D_2 = (A \to (B \to A)) \to (B \to (A \to A)) \\
\text{(C1)}: &\quad D_3 = A \to (B \to A) \\
\mathrm{MP}(D_3, D_2): &\quad D_4 = B \to (A \to A) \\
\mathrm{MP}(D_1, D_4): &\quad D_5 = A \to A
\end{aligned}$$

Example 2 $\quad \vdash (A \to (B \to C)) \to (B \to (A \to C))$

(**Exchange rule**, also called "exchange axiom")

$$
\begin{array}{rl}
\text{(A1) } A := B \wedge A, & \\
B := A \wedge B: & D_1 = (B \wedge A \to A \wedge B) \to ((A \wedge B \to C) \to (B \wedge A \to C)) \\
\text{(A3) } A :=: B: & D_2 = B \wedge A \to A \wedge B \\
\mathrm{MP}(D_2, D_3): & D_3 = 2\ (A \wedge B \to C) \to (B \wedge A \to C)\ 3 \\
\text{(A5a)}: & D_4 = 1\ (A \to (B \to C)) \to (A \wedge B \to C)\ 2 \\
\text{(A5b) } A :=: B: & D_5 = 3\ (B \wedge A \to C) \to (B \to (A \to C))\ 4 \\
\mathrm{TI}(D_4, D_3): & D_6 = (A \to (B \to C)) \to (B \wedge A \to C) \\
\mathrm{TI}(D_6, D_5): & D_7 = (A \to (B \to C)) \to (B \to (A \to C))
\end{array}
$$

$1 \to 2 \to 3 \to 4$

Example 2 $\vdash \boxed{(A \to (B \to C))} \to \boxed{(B \to (A \to C))}$

(**Exchange rule**, also called "exchange axiom")

$$
\begin{array}{rl}
\text{(A1) } A := \boxed{B \wedge A,} & \\
B := \boxed{A \wedge B} : & D_1 = \boxed{(B \wedge A \to A \wedge B)} \to ((A \wedge B \to C) \to (B \wedge A \to C)) \\
\text{(A3) } A :=: B : & D_2 = \boxed{B \wedge A \to A \wedge B} \\
\mathrm{MP}(D_2, D_3) : & D_3 \overset{2}{=} \boxed{(A \wedge B \to C)} \to \boxed{(B \wedge A \to C)} \;\; 3 \\
\text{(A5a)} : & D_4 \overset{1}{=} \boxed{(A \to (B \to C))} \to \boxed{(A \wedge B \to C)} \;\; 2 \\
\text{(A5b) } A :=: B : & D_5 \overset{3}{=} \boxed{(B \wedge A \to C)} \to \boxed{(B \to (A \to C))} \;\; 4 \\
\mathrm{TI}(D_4, D_3) : & D_6 = (A \to (B \to C)) \to (B \wedge A \to C) \\
\mathrm{TI}(D_6, D_5) : & D_7 = \boxed{(A \to (B \to C))} \to \boxed{(B \to (A \to C))}
\end{array}
$$

$1 \to 2 \to 3 \to 4$

Example 3 $\quad \vdash A \to A$

For brevity, let B denote a provable formula, e.g., axiom (A1).

$$\begin{aligned}
\text{(A1)}: &\quad D_1 = B \\
\text{Ex. 2}, C := A: &\quad D_2 = (A \to (B \to A)) \to (B \to (A \to A)) \\
\text{(C1)}: &\quad D_3 = A \to (B \to A) \\
\text{MP}(D_3, D_2): &\quad D_4 = B \to (A \to A) \\
\text{MP}(D_1, D_4): &\quad D_5 = A \to A
\end{aligned}$$

Example 3 $\quad \vdash A \to A$

For brevity, let B denote a provable formula, e.g., axiom (A1).

$$
\begin{array}{rl}
\text{(A1)}: & D_1 = \boxed{B} \\
\text{Ex. } 2, C := A: & D_2 = \boxed{(A \to (B \to A))} \to \boxed{(B \to (A \to A))} \\
\text{(C1)}: & D_3 = \boxed{A \to (B \to A)} \\
\text{MP}(D_3, D_2): & D_4 = \boxed{B} \to (\boxed{A \to A}) \\
\text{MP}(D_1, D_4): & D_5 = \boxed{A \to A}
\end{array}
$$