Representations of a fuzzy set

Horizontal representation: system of cuts
Vertical representation: membership function

Conversion from the horizontal to vertical representation:

$$\mu_A(x) = \sup\big\{\alpha \in [0,1] :\ x \in \mathcal{R}_A(\alpha)\big\}.$$

Theorem: (**the second representation theorem**) Let $A \in \mathcal{F}(X)$. Then

$$\mu_A = \sup_{\alpha \in [0,1]} \alpha\, \mu_{\mathcal{R}_A(\alpha)} = \sup_{\alpha \in \mathrm{Range}(A)} \alpha\, \mu_{\mathcal{R}_A(\alpha)},$$

where the supremum is computed pointwise, i.e.,

$$\mu_A(x) = \sup_{\alpha \in \mathrm{Range}(A)} \alpha\, \mu_{\mathcal{R}_A(\alpha)}(x).$$

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Classical definition

$A \subseteq B \iff \forall x \in A : \; x \in B$

cannot be used, because we cannot write $x \in A, x \in B$:

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For A, $B \in \mathcal{F}(X)$:
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$A \subseteq B \iff (\forall x \in X: \quad \mu_A(x) \le \mu_B(x)) \iff \mu_A \le \mu_B \iff$

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Proof of the last equivalence:

'$\Rightarrow$': Assume $\mu_A \leq \mu_B$, $x \in \mathcal{R}_A(\alpha)$,

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'$\Leftarrow$': Assume $\forall \alpha \in [0,1]: \ \mathcal{R}_A(\alpha) \subseteq \mathcal{R}_B(\alpha)$,

$$\mu_A(x) = \sup\{\alpha \in [0,1]: \ x \in \mathcal{R}_A(\alpha)\} \leq \sup\{\alpha \in [0,1]: \ x \in \mathcal{R}_B(\alpha)\} = \mu_B(x).$$

Cut-consistency

A **property** P of fuzzy sets $A_1, \dots, A_n$ maps arguments $A_1, \dots, A_n$ to a truth value $P(A_1, \dots, A_n) \in \{0, 1\}$ ("predicate").

Property P of of fuzzy sets is called

- **cutworthy** if

$$P(A_1, \dots, A_n) \Rightarrow (\forall \alpha \in (0,1] : P(\mathcal{R}_{A_1}(\alpha), \dots, \mathcal{R}_{A_n}(\alpha))) ,$$

- **cut-consistent** if

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(0-cuts are ignored intentionally)

Examples:

Inclusion is cut-consistent.

Strong normality, $\exists x \in X : \ \mu_A(x) = 1$, is cut-consistent.

Crispness is cutworthy, but not cut-consistent.

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Operations with crisp sets

set operations	propositional operations	formula
$\overline{} : \mathcal{P}(X) \to \mathcal{P}(X)$	$\neg : \{0,1\} \to \{0,1\}$	$\overline{A} = \big\{x \in X : \neg(x \in A)\big\}$
$\cap : \mathcal{P}(X)^2 \to \mathcal{P}(X)$	$\wedge : \{0,1\}^2 \to \{0,1\}$	$A \cap B = \big\{x \in X : (x \in A) \wedge (x \in B)\big\}$
$\cup : \mathcal{P}(X)^2 \to \mathcal{P}(X)$	$\vee : \{0,1\}^2 \to \{0,1\}$	$A \cup B = \big\{x \in X : (x \in A) \vee (x \in B)\big\}$

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By means of membership functions:

$\mu_{\overline{A}}(x) = \neg \mu_A(x)$
$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x)$
$\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x)$

Laws of Boolean algebras

$$
\begin{aligned}
\neg\neg\alpha &= \alpha, \\
\alpha \vee \beta &= \beta \vee \alpha, & \alpha \wedge \beta &= \beta \wedge \alpha, \\
(\alpha \vee \beta) \vee \gamma &= \alpha \vee (\beta \vee \gamma), & (\alpha \wedge \beta) \wedge \gamma &= \alpha \wedge (\beta \wedge \gamma), \\
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\alpha \vee \alpha &= \alpha, & \alpha \wedge \alpha &= \alpha, \\
\alpha \vee (\alpha \wedge \beta) &= \alpha, & \alpha \wedge (\alpha \vee \beta) &= \alpha, \\
\alpha \vee 1 &= 1, & \alpha \wedge 0 &= 0, \\
\alpha \vee 0 &= \alpha, & \alpha \wedge 1 &= \alpha, \\
\alpha \wedge \neg\alpha &= 0, & \alpha \vee \neg\alpha &= 1, \\
\neg(\alpha \wedge \beta) &= \neg\alpha \vee \neg\beta, & \neg(\alpha \vee \beta) &= \neg\alpha \wedge \neg\beta.
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Fuzzy negation

unary operation $\underset{\cdot}{\neg}\colon [0,1] \to [0,1]$ such that

$$\alpha \leq \beta \Rightarrow \underset{\cdot}{\neg}\beta \leq \underset{\cdot}{\neg}\alpha, \tag{N1}$$

$$\underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\alpha = \alpha. \tag{N2}$$

Example: **Standard negation**: $\underset{\mathrm{S}}{\neg}\alpha = 1 - \alpha$.

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Properties of fuzzy negations

Theorem: Each fuzzy negation $\underset{\cdot}{\neg}$ is a continuous, strictly decreasing bijection satisfying

$$\underset{\cdot}{\neg}\, 1 = 0, \qquad \underset{\cdot}{\neg}\, 0 = 1. \tag{N0}$$

Its graph is symmetric w.r.t. the axis of the 1st and 3rd quadrant, i.e., $\underset{\cdot}{\neg}^{-1} = \underset{\cdot}{\neg}$

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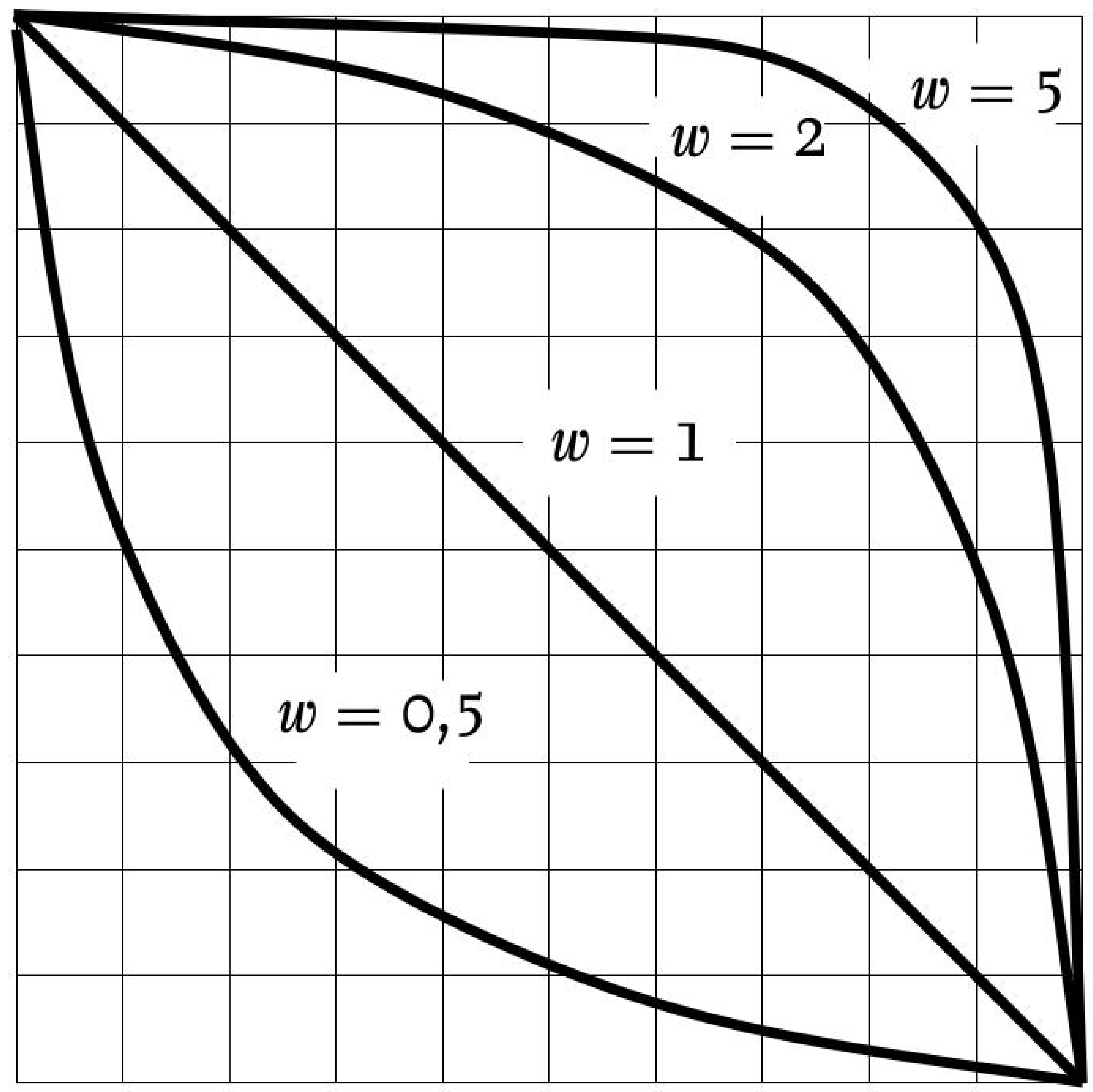
Proof:

- Injectivity: If $\underset{\cdot}{\neg}\,\alpha = \underset{\cdot}{\neg}\,\beta$, then $\alpha = \underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\alpha = \underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\,\beta = \beta$.
- Surjectivity: For each $\alpha \in [0,1]$ there is a $\beta \in [0,1]$ such that $\alpha = \underset{\cdot}{\neg}\,\beta$, namely $\beta = \underset{\cdot}{\neg}\,\alpha$.
- $\Rightarrow$ continuity and boundary conditions.
- The symmetry of the graph is equivalent to involutivity (N2).

Yager fuzzy negations

$$i(\alpha) = \alpha^w, \qquad i^{-1}(\alpha) = \alpha^{\frac{1}{w}}, \qquad \underset{\mathrm{Y_w}}{\neg}\,\alpha = i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big) = (1-\alpha^w)^{\frac{1}{w}}, \qquad w \in (0,\infty)$$



Representation theorem for fuzzy negations

A function $\underset{\cdot}{\neg}\colon [0,1] \to [0,1]$ is a fuzzy negation iff there is an increasing bijection $i\colon [0,1] \to [0,1]$ (**generator of fuzzy negation $\underset{\cdot}{\neg}$**) such that

$$\underset{\cdot}{\neg} = i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1}, \quad \text{i.e.,} \quad \underset{\cdot}{\neg}\,\alpha = i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big).$$

Proof:

- Sufficiency:

(N1): Assume $\alpha, \beta \in [0,1]$, $\alpha \le \beta$.
i, i^{-1} preserve the ordering, $\underset{\mathrm{S}}{\neg}$ reverses it:

$$\begin{aligned} i(\alpha) &\le i(\beta)\,, \\ \underset{\mathrm{S}}{\neg}\, i(\alpha) &\ge \underset{\mathrm{S}}{\neg}\, i(\beta)\,, \\ i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big) &\ge i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\beta)\big)\,, \\ \underset{\cdot}{\neg}\,\alpha &\ge \underset{\cdot}{\neg}\,\beta\,. \end{aligned}$$

(N2): $\underset{\cdot}{\neg} \circ \underset{\cdot}{\neg} = i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} \circ i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} = i \circ \underset{\mathrm{S}}{\neg} \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} = i \circ i^{-1} = \mathrm{id}$,
where id is the identity on $[0,1]$.

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Possible construction of a generator of a fuzzy negation

- Necessity (according to [Nguyen-Walker]): We shall prove that

$$i(\alpha) = \frac{\alpha + \underset{\mathrm{S}}{\neg} \underset{\cdot}{\neg} \alpha}{2}.$$

is a generator of a fuzzy negation $\underset{\cdot}{\neg}$.

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$$i^{-1}(\neg_S i(\alpha)) \overset{?}{=} \neg_. \alpha$$

$$L = \neg_S i(\alpha) \overset{?}{=} i(\neg_. \alpha) = R$$

$$L = 1 - i(\alpha) = \frac{2 - \alpha - (1 - \neg_. \alpha)}{2} = \frac{1 - \alpha + \neg_. \alpha}{2} =$$

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i is increasing, continuous, and satisfies $i(0) = 0$, $i(1) = 1$, thus i is a bijection on $[0, 1]$.

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A generator of a fuzzy negation is not unique.

Fuzzy complement

$$\mu_{\overline{A}^{\cdot}}(x) = \underset{\cdot}{\neg}\, \mu_A(x).$$

We distinguish them by the same indices as the corresponding fuzzy negations, e.g., $\overline{A}^S$ is the standard complement.

Fuzzy conjunction (triangular norm, t-norm)

binary operation $\underset{\cdot}{\wedge}\colon [0,1]^2 \to [0,1]$ such that, for all $\alpha, \beta, \gamma \in [0,1]$:

$\alpha \underset{\cdot}{\wedge} \beta = \beta \underset{\cdot}{\wedge} \alpha$	(commutativity)	(T1)
$\alpha \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\wedge} \gamma) = (\alpha \underset{\cdot}{\wedge} \beta) \underset{\cdot}{\wedge} \gamma$	(associativity)	(T2)
$\beta \leq \gamma \Rightarrow \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} \gamma$	(monotonicity)	(T3)
$\alpha \underset{\cdot}{\wedge} 1 = \alpha$	(boundary condition)	(T4)

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(N1): Assume $\alpha, \beta \in [0,1]$, $\alpha \le \beta$.
i, i^{-1} preserve the ordering, $\underset{S}{\neg}$ reverses it:

$$\begin{aligned} i(\alpha) &\le i(\beta)\,, \\ \underset{S}{\neg}\, i(\alpha) &\ge \underset{S}{\neg}\, i(\beta)\,, \\ i^{-1}\big(\underset{S}{\neg}\, i(\alpha)\big) &\ge i^{-1}\big(\underset{S}{\neg}\, i(\beta)\big)\,, \\ \underset{\cdot}{\neg}\,\alpha &\ge \underset{\cdot}{\neg}\,\beta\,. \end{aligned}$$

(N2): $\underset{\cdot}{\neg} \circ \underset{\cdot}{\neg} = i \circ \underset{S}{\neg} \circ i^{-1} \circ i \circ \underset{S}{\neg} \circ i^{-1} = i \circ \underset{S}{\neg} \circ \underset{S}{\neg} \circ i^{-1} = i \circ i^{-1} = \mathrm{id}$,
where id is the identity on $[0,1]$.

Representation theorem for fuzzy negations

A function $\underset{\cdot}{\neg}\colon [0,1] \to [0,1]$ is a fuzzy negation iff there is an increasing bijection $i\colon [0,1] \to [0,1]$ (**generator of fuzzy negation $\underset{\cdot}{\neg}$**) such that

$$\underset{\cdot}{\neg} = i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1}, \quad \text{i.e.,} \quad \underset{\cdot}{\neg}\, \alpha = i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big).$$

Proof:

- Sufficiency:

(N1): Assume $\alpha, \beta \in [0,1]$, $\alpha \le \beta$.
i, i^{-1} preserve the ordering, $\underset{\mathrm{S}}{\neg}$ reverses it:

$$\begin{aligned} i(\alpha) &\le i(\beta)\,, \\ \underset{\mathrm{S}}{\neg}\, i(\alpha) &\ge \underset{\mathrm{S}}{\neg}\, i(\beta)\,, \\ i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\alpha)\big) &\ge i^{-1}\big(\underset{\mathrm{S}}{\neg}\, i(\beta)\big)\,, \\ \underset{\cdot}{\neg}\, \alpha &\ge \underset{\cdot}{\neg}\, \beta\,. \end{aligned}$$

(N2): $\underset{\cdot}{\neg} \circ \underset{\cdot}{\neg} = i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} \circ i \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} = i \circ \underset{\mathrm{S}}{\neg} \circ \underset{\mathrm{S}}{\neg} \circ i^{-1} = i \circ i^{-1} = \mathrm{id}$,
where id is the identity on $[0,1]$.