From the deduction theorem in classical logic, only one direction holds in basic logic:

Theorem $\mathcal{T}$ ... theory

A, B ... formulas

$$\mathcal{T} \cup \{A\} \vdash B \Longleftarrow \mathcal{T} \vdash A \rightarrow B$$

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The proof is the same as in classical logic.

The other direction requires a weakening:

Theorem $\mathcal{T}$ ... theory

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$$\mathcal{T} \cup \{A\} \vdash B \Rightarrow \exists n \in N : (\mathcal{T} \vdash A^n \to B),$$

where $A^n = \underbrace{(A \wedge (A \wedge \cdots (A \wedge A) \cdots))}_{n\times}$

In view of (A5a), (A5b),

$$(A^n \to B) \leftrightarrow \underbrace{(A \to (A \to \cdots (A \to}_{n\times} B) \cdots))$$

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Proof Proof by contradiction:

Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$ and $\forall n \in N : (\mathcal{T} \nvdash A^n \to B)$.

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1. B is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

$$\begin{array}{rl} & D_1 = B \\ RI(D_1): & D_2 = A^n \to B \end{array}$$

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hence $\mathcal{T} \vdash A^n \to B$.

2. $B \neq A$ because $\vdash A^n \to A$ (without the need of $\mathcal{T}$):

for $n = 1$: $\vdash A \to A$

for $n > 1$: (A2) : $\vdash \underbrace{A \wedge A^{n-1}}_{A^n} \to A$

3. B is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.

WLOG, we choose for B a formula with the shortest possible proof; its shortest proof must be of the following form:

$$
\begin{array}{rl}
 & \vdots \\
 & D_i \\
 & \vdots \\
 & D_j = D_i \to B \\
 & \vdots \\
\text{MP}(D_i, D_j): & D_m = B
\end{array}
$$

for $i < j < m$ or $j < i < m$.

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for $i < j < m$ or $j < i < m$.

The proofs of $\mathcal{T} \cup \{A\} \vdash D_i$, $\mathcal{T} \cup \{A\} \vdash D_j$ are of lengths $< m$, therefore there are $n, k \in \mathbb{N}$ such that

$$\mathcal{T} \vdash A^n \to D_i$$
$$\mathcal{T} \vdash A^k \to D_j = A^k \to (D_i \to B)$$

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\begin{array}{l}
\mathcal{T} \vdash A^n \to D_i \\
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We shall prove $\mathcal{T} \vdash A^{n+k} \to B$:

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$\vdots$

$D_k = A^n \to D_i$

//END of proof of $\mathcal{T} \vdash A^n \to D_i$

//BEGIN of proof of $\mathcal{T} \vdash A^k \to D_j$

$\vdots$

$D_n = A^k \to \overbrace{(D_i \to B)}^{D_j}$

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$$
\begin{aligned}
\text{EA}(D_n): &\quad D_{n+1} = D_i \to (A^k \to B) \\
\text{TI}(D_k, D_{n+1}): &\quad D_{n+2} = A^n \to (A^k \to B) \\
\text{(A5a)}(D_{n+2}): &\quad D_{n+3} = A^{n+k} \to B
\end{aligned}
$$

We shall prove $\mathcal{T} \vdash A^{n+k} \to B$:

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⋮

$D_k = A^n \to D_i$

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⋮

$D_n = A^k \to \overbrace{(D_i \to B)}^{D_j}$

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EA(D_n) :	$D_{n+1} = D_i \to (A^k \to B)$
TI(D_k, D_{n+1}) :	$D_{n+2} = A^n \to (A^k \to B)$
(A5a)(D_{n+2}) :	$D_{n+3} = A^{n+k} \to B$

Deduction theorem in basic logic

$\mathcal{T}$ ... theory

A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\exists n \in N$: $(\mathcal{T} \vdash A^n \to B)$,
where $A^n = \underbrace{(A \wedge (A \wedge \cdots (A \wedge A) \cdots))}_{n\times}$

Example 4 $\vdash A \to (B \to A \land B)$

Example 4 $\vdash A \to (B \to A \wedge B)$

$$\begin{aligned} \text{(A3)}: &\quad D_1 = A \wedge B \to A \wedge B \\ \text{(A5b)}, C := A \wedge B: &\quad D_2 = (A \wedge B \to A \wedge B) \to (A \to (B \to A \wedge B)) \\ \text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A \wedge B) \end{aligned}$$

Corollary of the deduction theorem

$$A \vdash B \to A \wedge B$$
$$\{A, B\} \vdash A \wedge B$$

$$\begin{aligned} \text{(A2)}: &\quad A \wedge B \vdash A \\ \&\text{(A3)}: &\quad A \wedge B \vdash B \end{aligned}$$

Example 4 $\vdash A \to (B \to A \wedge B)$

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Corollary of the deduction theorem $A \vdash B \to A \wedge B$

$\{A, B\} \vdash A \wedge B$

$$\begin{aligned} \text{(A2)}: \quad & A \wedge B \vdash A \\ \&\text{(A3)}: \quad & A \wedge B \vdash B \end{aligned}$$

Corollary $\{A \to B, B \to A\} \vdash A \leftrightarrow B\,,$

$A \leftrightarrow B \vdash A \to B\,,$

$A \leftrightarrow B \vdash B \to A\,.$

Example 4 $\quad \vdash A \to (B \to A \wedge B)$

$$\begin{aligned}
\text{(A3)}: &\quad D_1 = A \wedge B \to A \wedge B \\
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\text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A \wedge B)
\end{aligned}$$

Corollary of the deduction theorem $\quad A \vdash B \to A \wedge B$

$$\{A, B\} \vdash A \wedge B$$

$$\begin{aligned}
\text{(A2)}: &\quad A \wedge B \vdash A \\
\&\text{(A3)}: &\quad A \wedge B \vdash B
\end{aligned}$$

Corollary $\quad \{A \to B, B \to A\} \vdash A \leftrightarrow B\,,$ $= (A \to B) \wedge (B \to A)$

$A \leftrightarrow B \vdash A \to B\,,$

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Example 4 $\vdash A \to (B \to A \wedge B)$

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Corollary of the deduction theorem $A \vdash B \to A \wedge B$

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Corollary $\{A \to B, B \to A\} \vdash A \leftrightarrow B$,
$A \leftrightarrow B \vdash A \to B$,
$A \leftrightarrow B \vdash B \to A$.

Relation $A \approx B$ iff $\vdash A \leftrightarrow B$ is an equivalence and $A \wedge B \approx B \wedge A$.

Example of deduction

$A \vdash B \to A \wedge (A \wedge B)$

Example of deduction $\quad A \vdash B \to A \wedge (A \wedge B)$

$$
\begin{aligned}
\text{SA}: &\quad D_1 = A \\
\text{Ex. 4, } B := A \wedge B: &\quad D_2 = A \to (A \wedge B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_2): &\quad D_3 = A \wedge B \to A \wedge (A \wedge B) \\
\text{(A5b)}, & \\
C := A \wedge (A \wedge B): &\quad D_4 = (A \wedge B \to A \wedge (A \wedge B)) \to (A \to (B \to A \wedge (A \wedge B))) \\
\text{MP}(D_3, D_4): &\quad D_5 = A \to (B \to A \wedge (A \wedge B)) \\
\text{MP}(D_1, D_5): &\quad D_6 = B \to A \wedge (A \wedge B)
\end{aligned}
$$

Here $\nvdash A \to (B \to A \wedge (A \wedge B))$
but $\vdash A \wedge A \to (B \to A \wedge (A \wedge B))$
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How can we simplify the proofs?

(A1) : $\vdash (A \to C) \to ((C \to B) \to (A \to B))$

Exchange rule: $\vdash (C \to B) \to ((A \to C) \to (A \to B))$

$\Downarrow$ (DT)

$C \to B \vdash (A \to C) \to (A \to B)$

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Example of deduction $\quad A \vdash B \to A \wedge (A \wedge B)$

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Theorem $C \to B \vdash (C \wedge A) \to (B \wedge A)$

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$$
\begin{aligned}
& D_1 = C \to B \\
\text{Ex. 4}: \quad & D_2 = B \to (A \to B \wedge A) \\
\text{(A1)}: \quad & D_3 = (C \to B) \to ((B \to (A \to B \wedge A)) \to (C \to (A \to B \wedge A))) \\
\text{MP}(D_1, D_3): \quad & D_4 = (B \to (A \to B \wedge A)) \to (C \to (A \to B \wedge A)) \\
\text{MP}(D_2, D_4): \quad & D_5 = C \to (A \to B \wedge A) \\
\text{(A5a)}: \quad & D_6 = (C \to (A \to B \wedge A)) \to (C \wedge A \to B \wedge A) \\
\text{MP}(D_5, D_6): \quad & D_7 = C \wedge A \to B \wedge A
\end{aligned}
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Theorem $C \to B \vdash (C \wedge A) \to (B \wedge A)$

	$D_1 = C \to B$
Ex. 4 :	$D_2 = B \to (A \to B \wedge A)$
(A1) :	$D_3 = (C \to B) \to ((B \to (A \to B \wedge A)) \to (C \to (A \to B \wedge A)))$
MP(D_1, D_3) :	$D_4 = (B \to (A \to B \wedge A)) \to (C \to (A \to B \wedge A))$
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B :=: C: \quad & B \to C \vdash (B \wedge A) \to (C \wedge A) \\
& B \leftrightarrow C \vdash (B \wedge A) \leftrightarrow (C \wedge A) \\
& B \approx C \Rightarrow (B \wedge A) \approx (C \wedge A) \\
\&\text{(A3)}: \quad & B \approx C \Rightarrow (A \wedge B) \approx (A \wedge C)
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Theorem $C \to B \vdash (C \wedge A) \to (B \wedge A)$

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Substitutions by equivalent formulas can be applied locally to any subformulas.

INTERPLAY OF SYNTAX AND SEMANTICS OF BASIC LOGIC

Soundness Each provable formula is a 1-tautology, i.e., if $\vdash A$, then $\models A$. Moreover, for any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness Each 1-tautology is provable, i.e., if $\models A$, then $\vdash A$.

Strong completeness [Hájek 1998]
For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$.
(We consider all evaluations with values in BL-algebras.)

Standard completeness [Cignoli, R., Esteva, F., Godo, L., Torrens, A. 2000]
Each formula which is evaluated to 1 by all **standard** evaluations (with values in $[0,1]$ and an arbitrary continuous fuzzy conjunction) is provable.

Example $\nvdash (A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

Not valid for Łukasiewicz operations and A, B, C evaluated to $1/2$.

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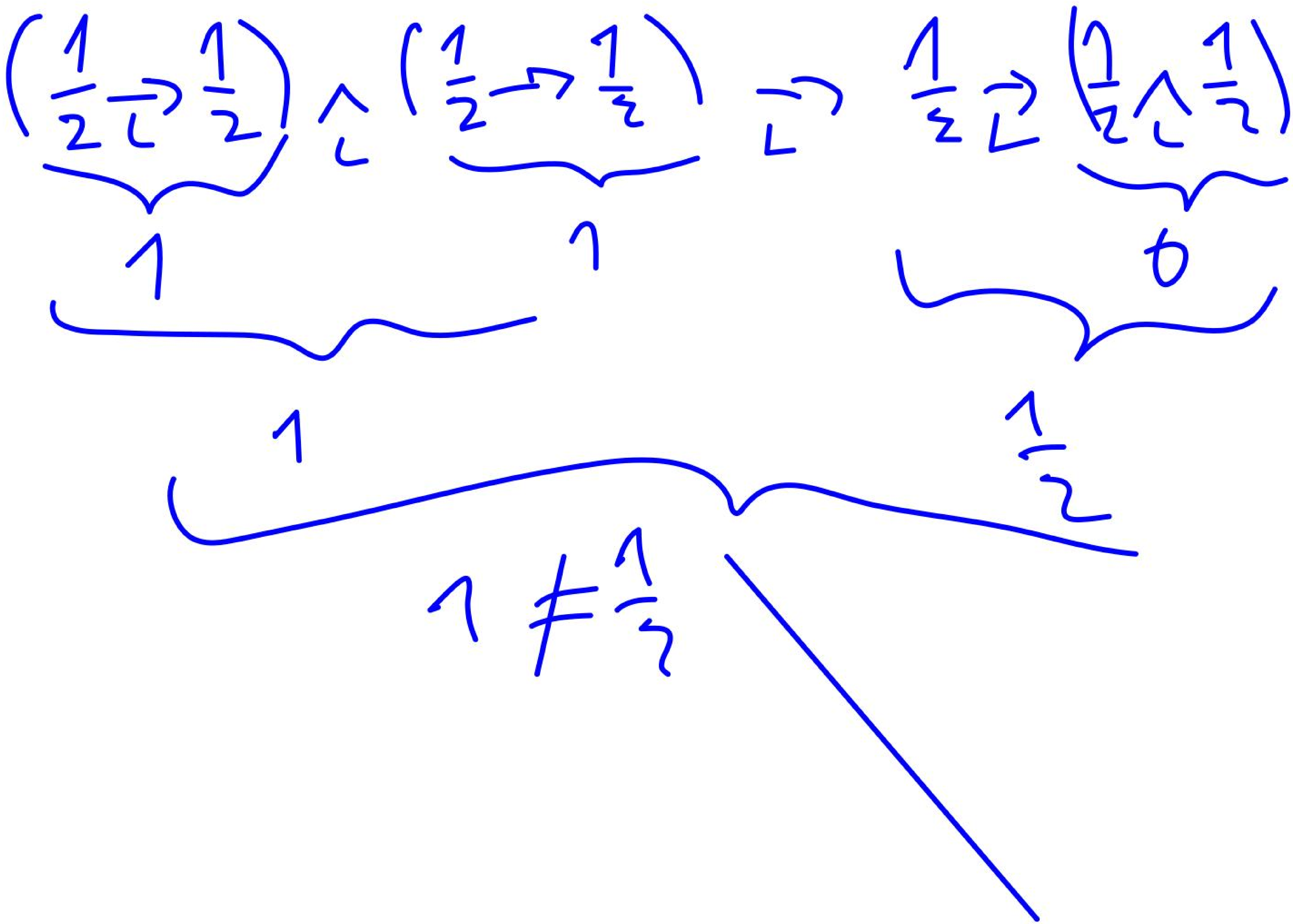

Example $\nvdash (A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

Not valid for Łukasiewicz operations and A, B, C evaluated to $1/2$.

Exercise Which axioms of the classical logic are 1-tautologies of BL (and hence provable in BL)?

How do the properties of interpretation of conjunction (fuzzy conjunction) follow from logical axioms?

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Commutativity of $\dot{\wedge}$ follows directly from (A3).

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$$(A \wedge B) \leftrightarrow (B \wedge A)$$

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Associativity of $\underset{\cdot}{\wedge}$:

TFAE (A5):

$(A \wedge B) \wedge C \to D$

$A \wedge B \to (C \to D)$

$A \to (B \to (C \to D))$

Now we use the equivalence of subformulas:

$B \to (C \to D)$

$B \wedge C \to D$

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TFAE (A5): $\leq \delta$

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$\leq \delta$

Associativity of $\land$:

TFAE (A5):

$(A \land B) \land C \to D$

$A \land B \to (C \to D)$

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    $B \to (C \to D)$

    $B \land C \to D$

$A \to (B \land C \to D)$

$A \land (B \land C) \to D$

We proved:

$((A \land B) \land C \to D) \leftrightarrow (A \land (B \land C) \to D)$

for all D, in particular, for $D := (A \land B) \land C$:

$((A \land B) \land C \to (A \land B) \land C) \leftrightarrow$

$(A \land (B \land C) \to (A \land B) \land C)$

MP: $\quad A \land (B \land C) \to (A \land B) \land C$

and for $D := A \land (B \land C)$ the other implication, hence

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GÖDEL LOGIC

Standard semantics:

$\wedge$ ... idempotent=standard fuzzy conjunction, $\underset{S}{\wedge} = \min$

$\rightarrow$ ... residuum of $\underset{S}{\wedge}$, Gödel implication $\underset{S}{\rightarrow}$

$\neg$ ... Gödel generalized negation $\underset{G}{\neg}$:

$$\underset{G}{\neg} x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

There is no connective in Gödel logic which is interpreted by the standard fuzzy negation.

Syntax:

Axioms (A1)–(A7) and

(G) $A \rightarrow A \wedge A$

Corollary: $A \leftrightarrow A \wedge A$

Deduction rule: Modus Ponens

Only in Gödel logic the classical deduction theorem holds (because $A^n \leftrightarrow A$):

Deduction theorem in Gödel logic

$\mathcal{T}$ ... a finite theory

A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \rightarrow B$

Standard completeness of Gödel logic

$(\vdash A) \iff (\models A)$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0,1]$ with Gödel operations).

Moreover, for any finite theory $\mathcal{T}$, $\qquad (\mathcal{T} \vdash A) \iff (\mathcal{T} \models A)$.

$$\begin{array}{ccc}
\text{theorem of CL} & \iff & \text{tautology of CL} \\
\Uparrow & & \Uparrow \\
\text{theorem of GL} & \iff & \text{1-tautology of GL} \\
\Uparrow & & \Uparrow \\
\text{theorem of BL} & \iff & \text{1-tautology of BL}
\end{array}$$