Fuzzy Sets

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http://cmp.felk.cvut.cz/~navara/fl/fset_printE.pdf

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Minimum about (classical) sets

To avoid problems of the set theory, we restrict ourselves to subsets of some **universal set** (**universe**) X

$\mathcal{P}(X)$ denotes the set of all subsets of a set X

A set $A \in \mathcal{P}(X)$ is uniquely determined by its **characteristic function** (**indicator**) $\mu_A : X \to \{0,1\}$,

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

$$A = \{x \in X :\ \mu_A(x) = 1\} = \{x \in X :\ \mu_A(x) > 0\}.$$

Using the notation

$$\mu_A^{-1}(M) = \{x \in X :\ \mu_A(x) \in M\},$$

we may write

$$A = \mu_A^{-1}\big(\{1\}\big) = \mu_A^{-1}\big((0,1]\big).$$

Instead of $\mu_A^{-1}\big(\{1\}\big)$, we write $\mu_A^{-1}(1)$, etc.

In particular $\mu_\emptyset = 0$, $\mu_X = 1$.

Definition of fuzzy sets

A **fuzzy subset** of a universe X (a **fuzzy set**) is a mathematical object A described by its (generalized) **characteristic function** (**membership function**) $\mu_A : X \to [0,1]$

Alternative notation: $A(x)$

In this context, "classical" sets are called **crisp** or **sharp**.

$\mathcal{F}(X)$ denotes the set of all fuzzy subsets of a universe X

Range: $\mathrm{Range}(A) = \big\{\alpha \in [0,1] :\ (\exists x \in X :\ \mu_A(x) = \alpha)\big\} = \mu_A(X)$

Height: $h(A) = \sup \mathrm{Range}(A)$

Support: $\mathrm{Supp}(A) = \big\{x \in X :\ \mu_A(x) > 0\big\} = \mu_A^{-1}\big((0,1]\big)$

Core: $\mathrm{core}(A) = \big\{x \in X :\ \mu_A(x) = 1\big\} = \mu_A^{-1}(1)$

Examples of fuzzy sets

$A, B \in \mathcal{F}(\mathbb{R})$,

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \in [0,1], \\ 2-x & \text{if } x \in (1,2], \\ 0 & \text{if } x > 2, \end{cases}$$

$$\mu_B(x) = \begin{cases} \frac{1}{2} & \text{if } x = 3, \\ 1 & \text{if } x = 4, \\ \frac{1}{4} & \text{if } x = 5, \\ 0 & \text{otherwise.} \end{cases}$$

For finite fuzzy sets, we use an abbreviated notation like $\mu_B = \{(3, \frac{1}{2}), (4, 1), (5, \frac{1}{4})\}$.

Alternative notations: $\mu_B = \{\frac{1}{2}/3, 1/4, \frac{1}{4}/5\}$, $\mu_B = \frac{1}{2}/3 + 1/4 + \frac{1}{4}/5$.

System of cuts of a fuzzy set

Definition: Let $A \in \mathcal{F}(X)$, $\alpha \in [0,1]$. The **α-level** of A is the crisp set

$$\mu_A^{-1}(\alpha) = \{x \in X : \mu_A(x) = \alpha\}.$$

The **system of cuts** of A is the mapping $\mathcal{R}_A : [0,1] \to \mathcal{P}(X)$ which assigns to each $\alpha \in [0,1]$ the **α-cut**

$$\mathcal{R}_A(\alpha) = \mu_A^{-1}\big([\alpha, 1]\big) = \{x \in X : \mu_A(x) \geq \alpha\}.$$

The **system of strong cuts** is the mapping $\mathcal{S}_A : [0,1] \to \mathcal{P}(X)$, where

$$\mathcal{S}_A(\alpha) = \mu_A^{-1}\big((\alpha, 1]\big) = \{x \in X : \mu_A(x) > \alpha\}.$$

Alternative notations of α-cuts: $[A]_\alpha$, $[A]^\alpha$, ${}^\alpha A$, ${}_\alpha A$

$$
\begin{aligned}
\mathrm{Range}(A) &= \{\alpha \in [0,1] :\ \mu_A^{-1}(\alpha) \neq \emptyset\},\\
h(A) &= \sup\{\alpha \in [0,1] :\ \mathcal{R}_A(\alpha) \neq \emptyset\},\\
\mathrm{Supp}(A) &= \mathcal{S}_A(0),\\
\mathrm{core}(A) &= \mathcal{R}_A(1),\\
\mathcal{R}_A(0) &= X,\\
\mathcal{S}_A(1) &= \emptyset.
\end{aligned}
$$

The first representation theorem

Theorem: A mapping $M : [0,1] \to \mathcal{P}(X)$ is the system of cuts of some fuzzy set $A \in \mathcal{F}(X)$ if and only if

(R1) $M(0) = X$,

(R2) $0 \leq \alpha < \beta \leq 1 \Rightarrow M(\alpha) \supseteq M(\beta)$,

(R3) $0 < \beta \leq 1 \Rightarrow M(\beta) = \bigcap_{\alpha:\alpha<\beta} M(\alpha)$.

Proof:

'$\Rightarrow$' ("$\mathcal{R}_A = M$ satisfies (1)–(3)"):

(R1): $M(0) = \mathcal{R}_A(0) = X$.

(R2): $x \in M(\beta) = \mathcal{R}_A(\beta) \Rightarrow \mu_A(x) \geq \beta > \alpha \Rightarrow x \in \mathcal{R}_A(\alpha) = M(\alpha)$.

(R3) '$\subseteq$': (R2) $\Rightarrow \forall \alpha \in [0,\beta) : M(\beta) \subseteq M(\alpha) \Rightarrow M(\beta) \subseteq \bigcap_{\alpha:\alpha<\beta} M(\alpha)$.

(R3) '$\supseteq$': $x \in \bigcap_{\alpha:\alpha<\beta} M(\alpha) = \bigcap_{\alpha:\alpha<\beta} \mathcal{R}_A(\alpha) \Rightarrow \forall \alpha \in [0,\beta) : \mu_A(x) \geq \alpha$,

$$\Rightarrow \mu_A(x) \geq \beta \iff x \in \mathcal{R}_A(\beta) = M(\beta).$$

'$\Leftarrow$' ("from M satisfying (1)–(3) to $\mathcal{R}_A$"):

We shall prove that $M = \mathcal{R}_A$, where $\mu_A(x) := \sup\{\alpha \in [0,1] : \ x \in M(\alpha)\}$.

'$\subseteq$': $x \in M(\beta) \Rightarrow \mu_A(x) \geq \beta \iff x \in \mathcal{R}_A(\beta)$,

'$\supseteq$': $x \in \mathcal{R}_A(\beta) \Rightarrow \mu_A(x) = \sup\{\alpha \in [0,1] : \ x \in M(\alpha)\} \geq \beta$,

$\forall \alpha \in [0,\beta) : x \in M(\alpha)$,

$x \in \bigcap_{\alpha:\alpha<\beta} M(\alpha) = M(\beta)$.

Representations of a fuzzy set

Horizontal representation: system of cuts
Vertical representation: membership function

Conversion from the horizontal to vertical representation:

$$\mu_A(x) = \sup\big\{\alpha \in [0,1] :\ x \in \mathcal{R}_A(\alpha)\big\}.$$

Theorem: (**the second representation theorem**) Let $A \in \mathcal{F}(X)$. Then

$$\mu_A = \sup_{\alpha \in [0,1]} \alpha\, \mu_{\mathcal{R}_A(\alpha)} = \sup_{\alpha \in \mathrm{Range}(A)} \alpha\, \mu_{\mathcal{R}_A(\alpha)},$$

where the supremum is computed pointwise, i.e.,

$$\mu_A(x) = \sup_{\alpha \in \mathrm{Range}(A)} \alpha\, \mu_{\mathcal{R}_A(\alpha)}(x).$$

Fuzzy inclusion

Classical definition
$A \subseteq B \iff \forall x \in A: \;\; x \in B$
cannot be used, because we cannot write $x \in A, x \in B$

However, we can write
$A \subseteq B \iff \forall x \in X: \;\; \mu_A(x) \leq \mu_B(x) \iff \mu_A \leq \mu_B$

For A, $B \in \mathcal{F}(X)$:
$A \subseteq B \iff \forall x \in X: \;\; \mu_A(x) \leq \mu_B(x) \iff \mu_A \leq \mu_B \iff$
$\forall \alpha \in [0,1]: \;\; \mathcal{R}_A(\alpha) \subseteq \mathcal{R}_B(\alpha)$

Proof of the last equivalence:

'$\Rightarrow$': Assume $\mu_A \leq \mu_B$, $x \in \mathcal{R}_A(\alpha)$,

$\alpha \leq \mu_A(x) \leq \mu_B(x)$, $x \in \mathcal{R}_B(\alpha)$, i.e., $\mathcal{R}_A(\alpha) \subseteq \mathcal{R}_B(\alpha)$

'$\Leftarrow$': Assume $\forall \alpha \in [0,1]: \;\; \mathcal{R}_A(\alpha) \subseteq \mathcal{R}_B(\alpha)$,

$$\mu_A(x) = \sup\{\alpha \in [0,1]: \; x \in \mathcal{R}_A(\alpha)\} \leq \sup\{\alpha \in [0,1]: \; x \in \mathcal{R}_B(\alpha)\} = \mu_B(x)$$

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Cut-consistency

A **property** P of fuzzy sets $A_1, \ldots, A_n$ maps arguments $A_1, \ldots, A_n$ to a truth value $P(A_1, \ldots, A_n) \in \{0, 1\}$ ("predicate").

Property P of of fuzzy sets is called

- **cutworthy** if

$$P(A_1, \ldots, A_n) \Rightarrow (\forall \alpha \in (0, 1] : P(\mathcal{R}_{A_1}(\alpha), \ldots, \mathcal{R}_{A_n}(\alpha))) \,,$$

- **cut-consistent** if

$$P(A_1, \ldots, A_n) \Longleftrightarrow (\forall \alpha \in (0, 1] : P(\mathcal{R}_{A_1}(\alpha), \ldots, \mathcal{R}_{A_n}(\alpha))) \,.$$

(0-cuts are ignored intentionally)

Examples:

Inclusion is cut-consistent.

Strong normality, $\exists x \in X : \ \mu_A(x) = 1$, is cut-consistent.

Crispness is cutworthy, but not cut-consistent.