

COMPACTNESS OF LOGICS

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In classical logic, an equivalent formulation is:

Compactness Theorem II: A formula is provable in a theory iff it is provable in some its **finite** subtheory.

Compactness Theorem II holds in classical, basic, Gödel, Łukasiewicz, and product logic, but **not** in RPL.

Example:

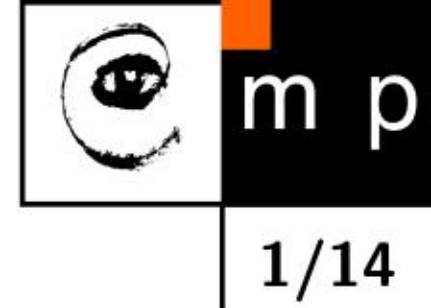
$\forall n \in \mathbb{N} : r_n := 1 - \frac{1}{n}$, A formula which is not provable (e.g., a variable),

$\mathcal{T} = \{r_n \rightarrow A : n \in \mathbb{N}\}$,

$|A|_{\mathcal{T}} \geq \sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$,

but A cannot be proved from any finite subtheory.

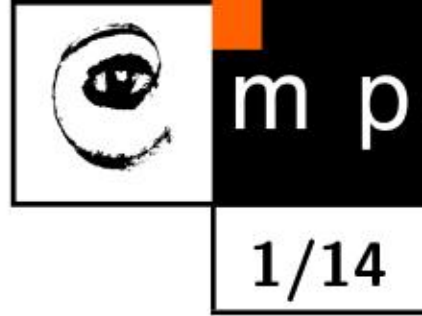
Center for Machine Perception presents



Mirko Navara (Praha)

Semantical testing of tautologies in many-valued logics

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What can computers do for us?

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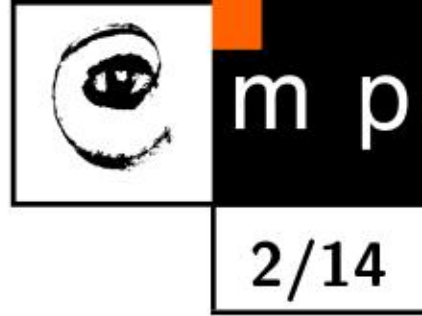
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Semantical testing of tautologies in many-valued logics

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Semantical testing of tautologies



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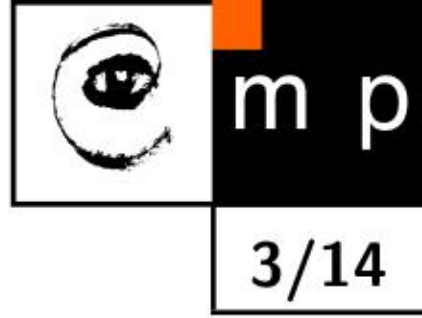
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In **many-valued logics**:

Depends on the choice of many-valued logic;

the most interesting progress has been made in the Łukasiewicz logic, i.e., in MV-algebras

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M	number of truth values-1
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3	68 719 476 736
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$M \backslash n$	1	2	3
1	152		
2	2147 581 952	93 831 434 829 824	
3	$2.361 \cdot 10^{21}$	$1.081 \cdot 10^{32}$	$5.575 \cdot 10^{42}$

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Complexity: $(b_1(M) + 1)^n$

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1	2				
2	3	9			
3	5	25	125		
4	9	81	729	6561	
5	17	289	4913	83 521	1419 857
6	33	1089	35 937	1185 921	39 135 393
7	65	4225	274 625	17 850 625	1160 290 625

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Implemented by [Brůžková 05].

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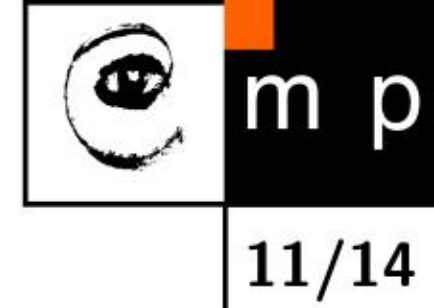
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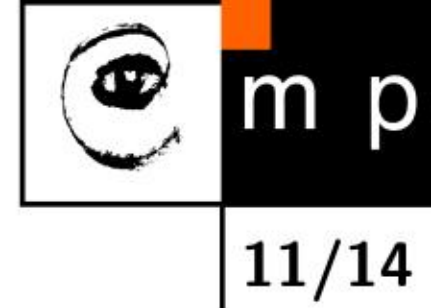
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Semantical testing in many-valued logics 2



Related questions:

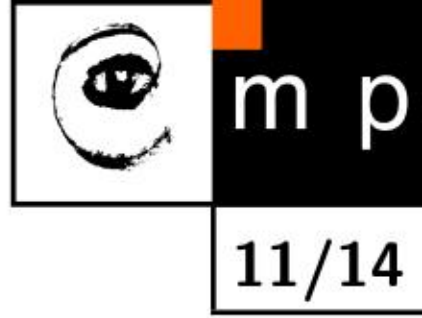
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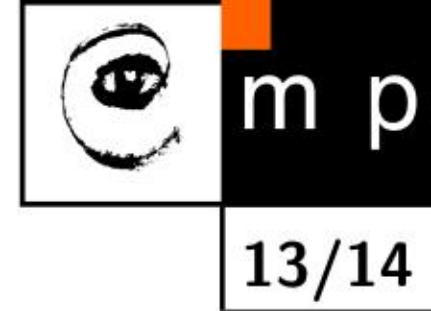
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- Testing of tautologies in basic logic?

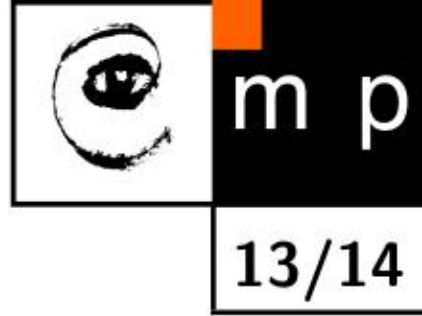
[Hájek; Haniková; Montagna, Pinna, and Tiezzi 03]; so far no implementation.

Semantical testing in many-valued logics 3



Alternative approaches to testing of tautologies:

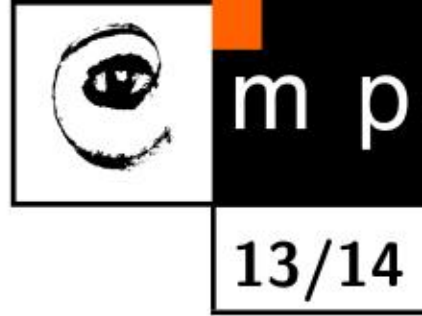
Semantical testing in many-valued logics 3



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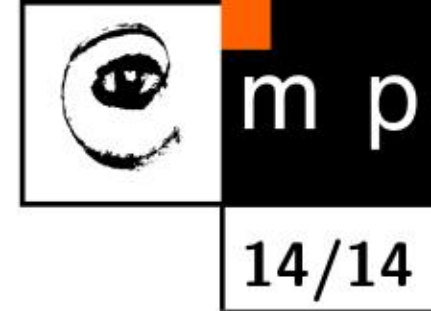
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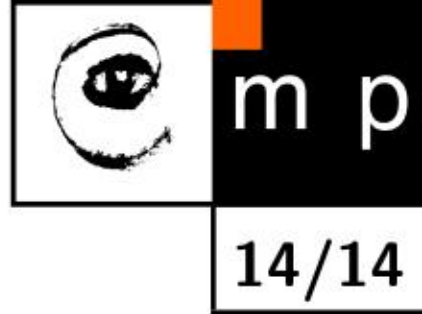
Programmed by [Hähnle et al. ~95].

Semantical testing in many-valued logics 4



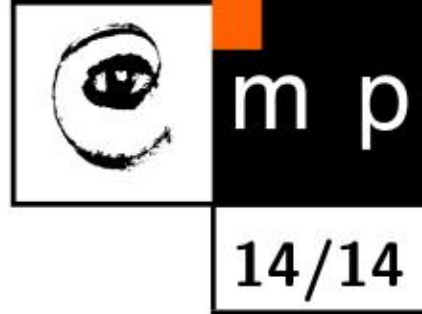
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Semantical testing in many-valued logics 4



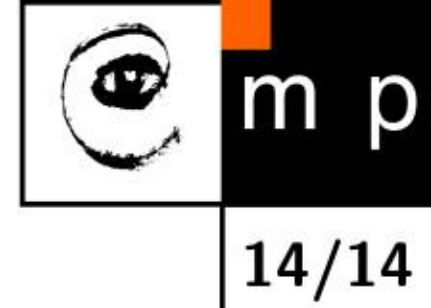
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A chance to obtain a **positive answer**.

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May give only a **negative answer**.

- Syntactical prover [Lehmke 05] <http://ls1-www.cs.uni-dortmund.de/~lehmke/SimpleProver>

Normally, the length of proofs is at most 10, but with a heuristic search, a proof of length of 18 has been obtained.

It proved the dependence of the axioms A2 and A3 of the Hájek's basic logic.

A chance to obtain a **positive answer**.

The latter two methods do not guarantee an ultimate answer, but they give a reasonable chance to obtain it.