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In classical logic, an equivalent formulation is:
Compactness Theorem II: A formula is provable in a theory iff it is provable in some its **finite** subtheory.

Compactness Theorem II holds in classical, basic, Gödel, Łukasiewicz, and product logic, but **not** in RPL.

Example:
$\forall n \in \mathbb{N} : r_n := 1 - \frac{1}{n}$, A formula which is not provable (e.g., a variable),
$\mathcal{T} = \{\mathbf{r}_n \to A : n \in \mathbb{N}\}$,
$|A|_{\mathcal{T}} \geq \sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$,
but A cannot be proved from any finite subtheory.

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Semantical testing of tautologies in many-valued logics

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What can computers do for us?

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What can computers do for us?

(And what they cannot do.)

Semantical testing of tautologies

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In **many-valued logics**:

Depends on the choice of many-valued logic;
the most interesting progress has been made in the Łukasiewicz logic, i.e., in MV-algebras

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$M \backslash n$	1	2	3
1	152		
2	2147 581 952	93 831 434 829 824	
3	$2.361 \cdot 10^{21}$	$1.081 \cdot 10^{32}$	$5.575 \cdot 10^{42}$

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Complexity: $(b_1(M)+1)^n$

$M \backslash n$	1	2	3	4	5
1	2				
2	3	9			
3	5	25	125		
4	9	81	729	6561	
5	17	289	4913	83 521	1419 857
6	33	1089	35 937	1185 921	39 135 393
7	65	4225	274 625	17 850 625	1160 290 625

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...
- Testing of tautologies in basic logic?
[Hájek; Haniková; Montagna, Pinna, and Tiezzi 03]; so far no implementation.

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Programmed by [Hähnle et al. ~95].

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The latter two methods do not guarantee an ultimate answer, but they give a reasonable chance to obtain it.