Fuzzy conjunction (triangular norm, t-norm)

binary operation $\underset{\cdot}{\wedge} : [0,1]^2 \to [0,1]$ such that, for all $\alpha, \beta, \gamma \in [0,1]$:

$\alpha \underset{\cdot}{\wedge} \beta = \beta \underset{\cdot}{\wedge} \alpha$	(commutativity)	(T1)
$\alpha \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\wedge} \gamma) = (\alpha \underset{\cdot}{\wedge} \beta) \underset{\cdot}{\wedge} \gamma$	(associativity)	(T2)
$\beta \leq \gamma \Rightarrow \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} \gamma$	(monotonicity)	(T3)
$\alpha \underset{\cdot}{\wedge} 1 = \alpha$	(boundary condition)	(T4)

Theorem: $\alpha \underset{\cdot}{\wedge} 0 = 0.$

Proof: Using (T3) and (T4): $\alpha \underset{\cdot}{\wedge} 0 \overset{(T3)}{\leq} 1 \underset{\cdot}{\wedge} 0 \overset{(T4)}{=} 0.$

Examples of fuzzy conjunctions

- **Standard** conjunction (**min**, **Gödel**, **Zadeh**, . . .):

$$\alpha \underset{\mathrm{S}}{\wedge} \beta = \min(\alpha, \beta).$$

- **Łukasiewicz** conjunction (**Giles**, **bold**, . . .):

$$\alpha \underset{\mathrm{L}}{\wedge} \beta = \begin{cases} \alpha + \beta - 1 & \text{if } \alpha + \beta - 1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- **Product** conjunction (**probabilistic**, **Goguen**, **algebraic product**, . . .):

$$\alpha \underset{\mathrm{P}}{\wedge} \beta = \alpha \cdot \beta.$$

- **Drastic** conjunction (**weak**, . . .):

$$\alpha \underset{\mathrm{D}}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1, \\ \beta & \text{if } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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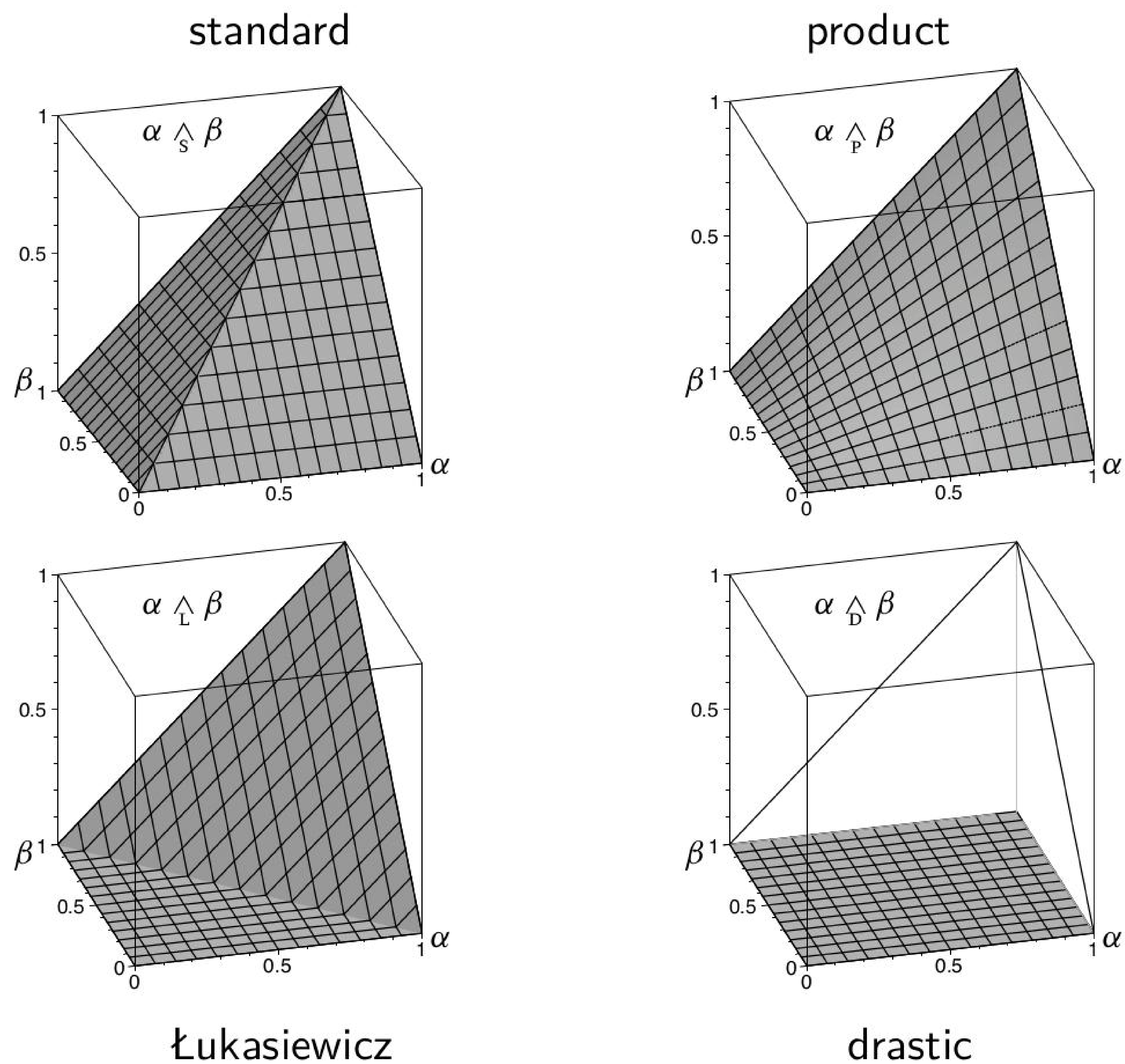

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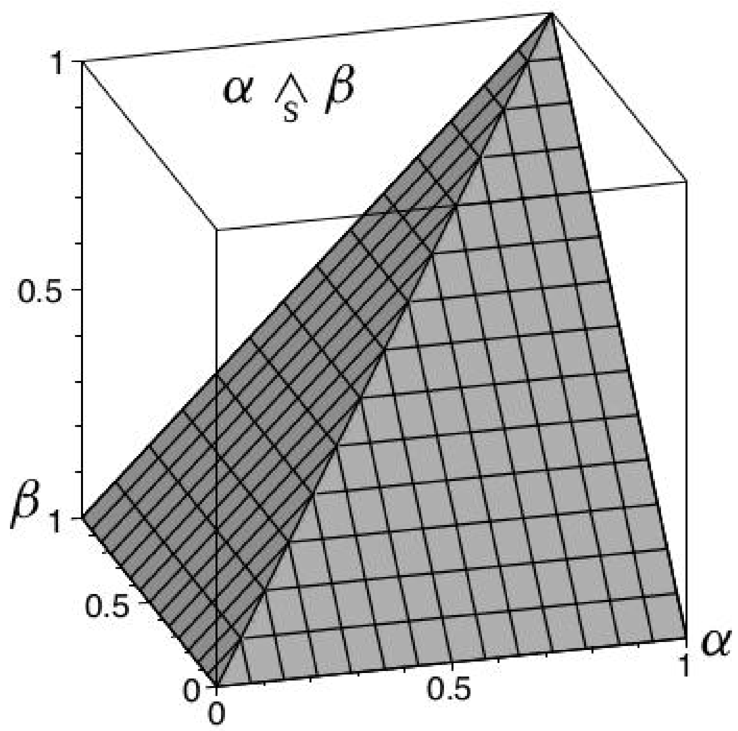
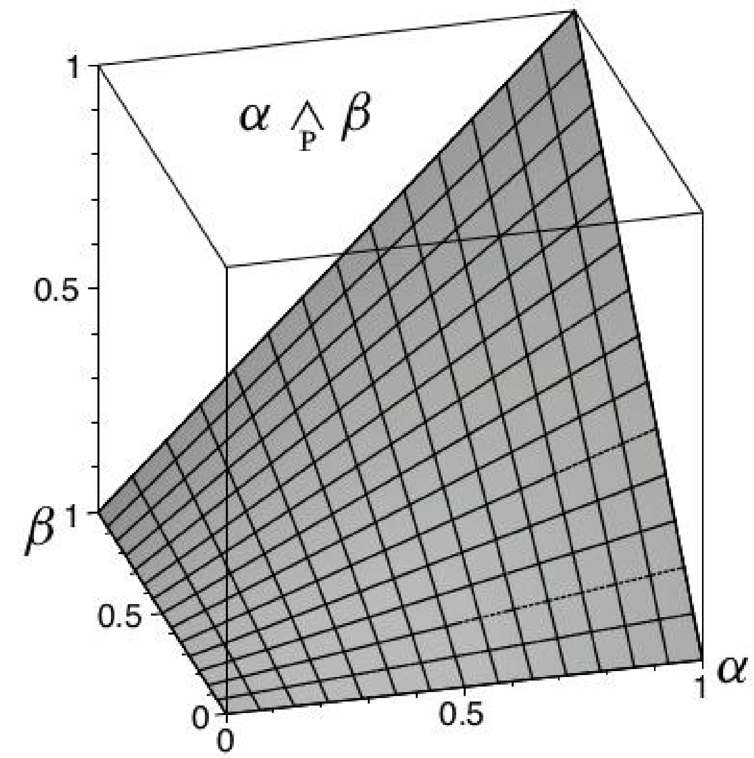
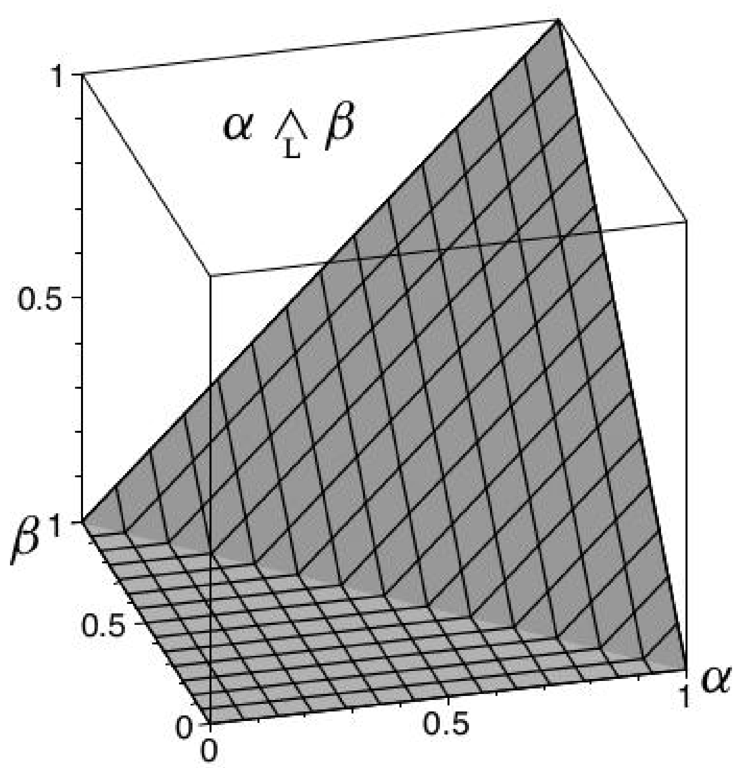
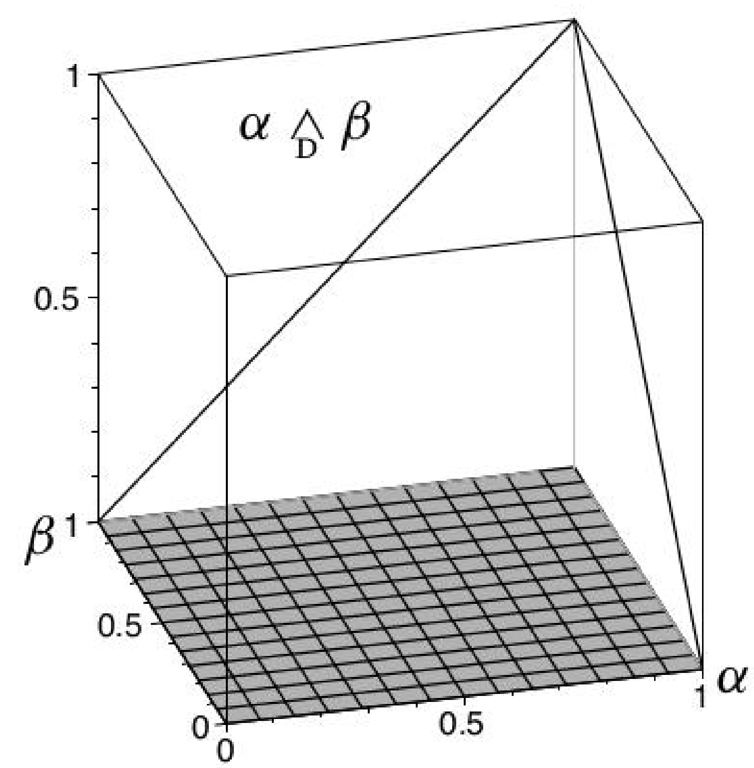


product





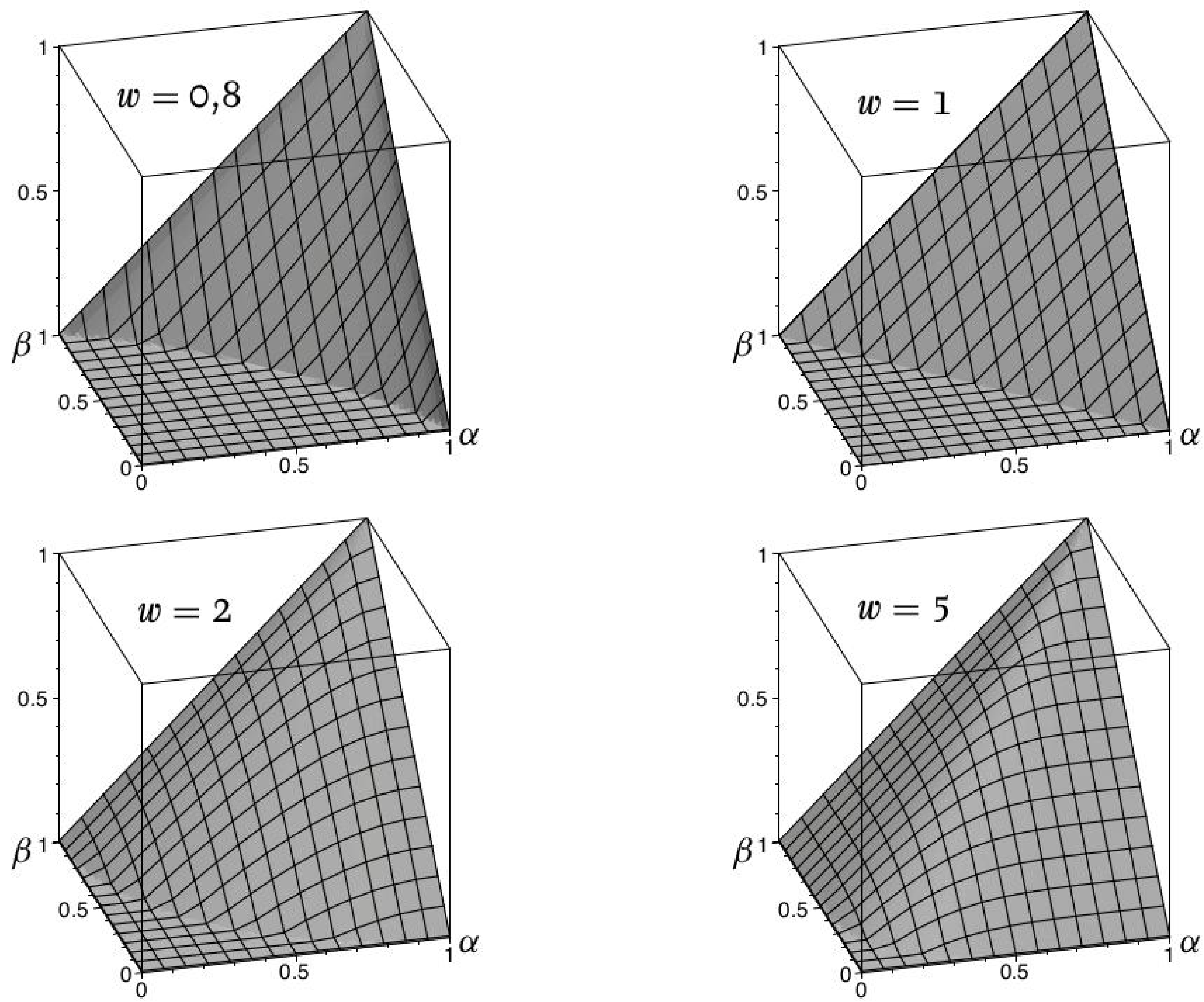
Łukasiewicz



drastic

Yager fuzzy conjunctions

$$\alpha \underset{\mathrm{Y_w}}{\wedge} \beta = \max\left(1 - \left((\alpha-1)^w + (\beta-1)^w\right)^{\frac{1}{w}}, 0\right)$$



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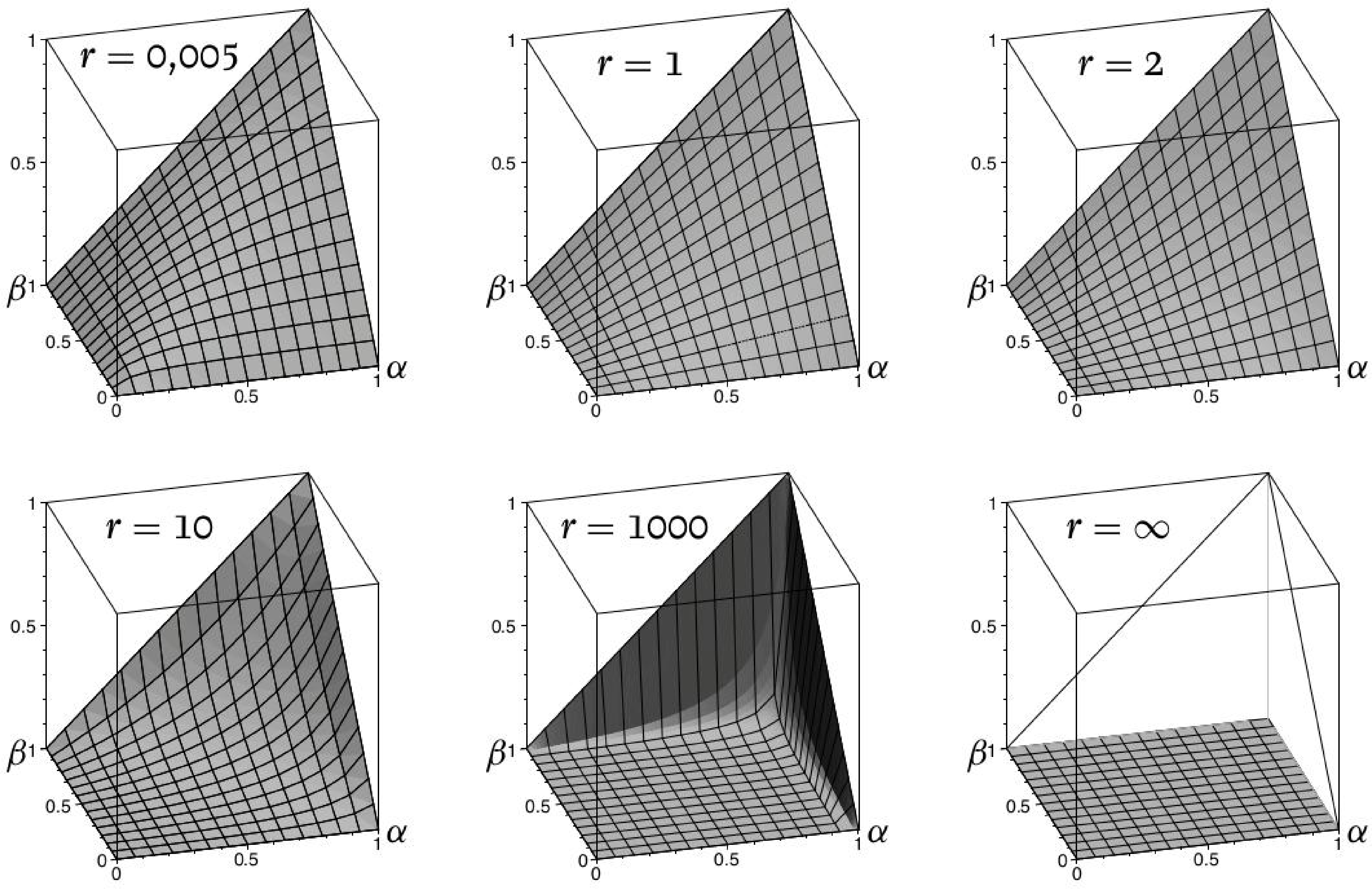
$$\alpha \underset{\mathrm{H_r}}{\wedge} \beta = \frac{\alpha\beta}{r + (1-r)\,(\alpha + \beta - \alpha\beta)}$$



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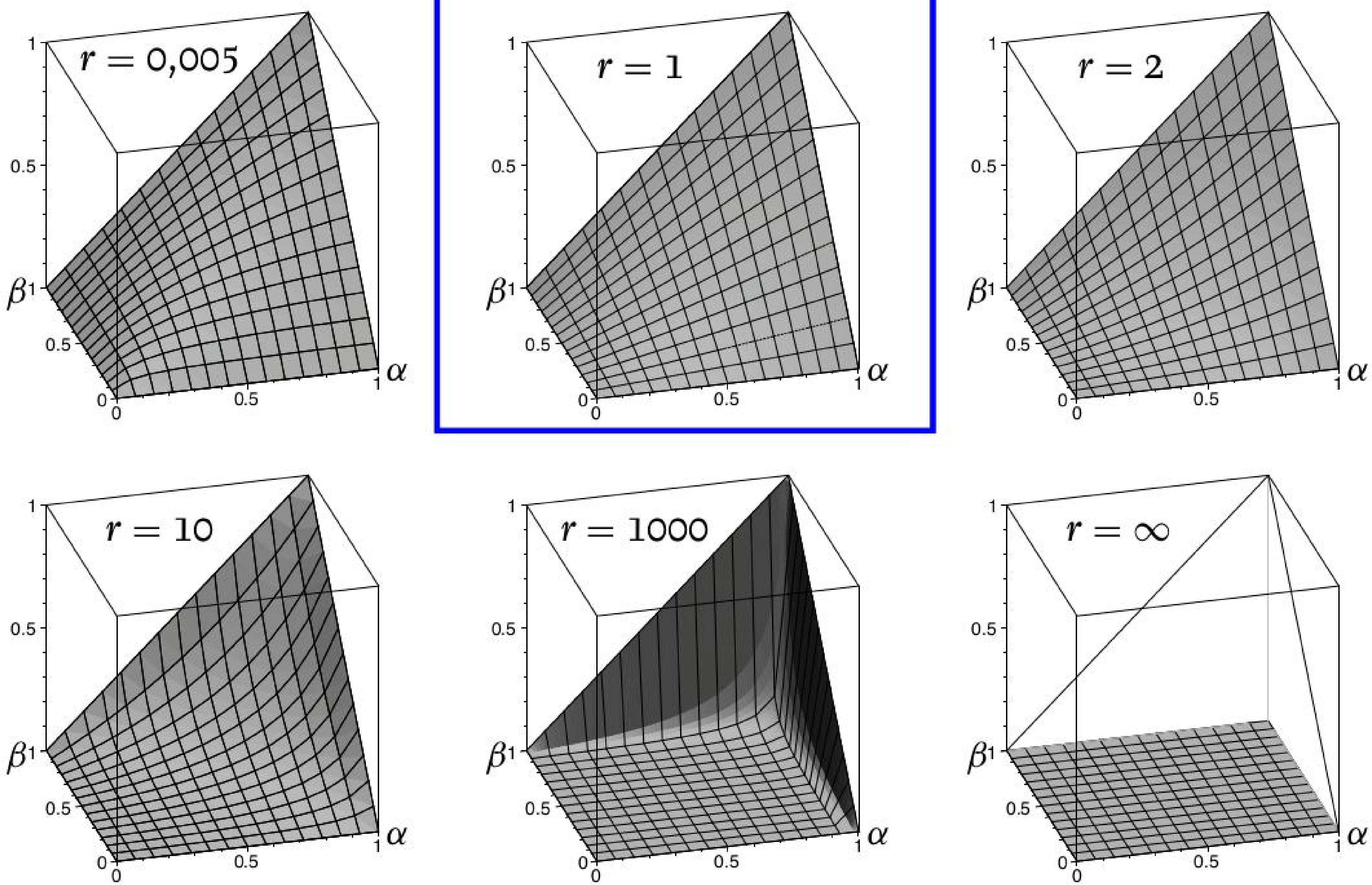

Properties of fuzzy conjunctions

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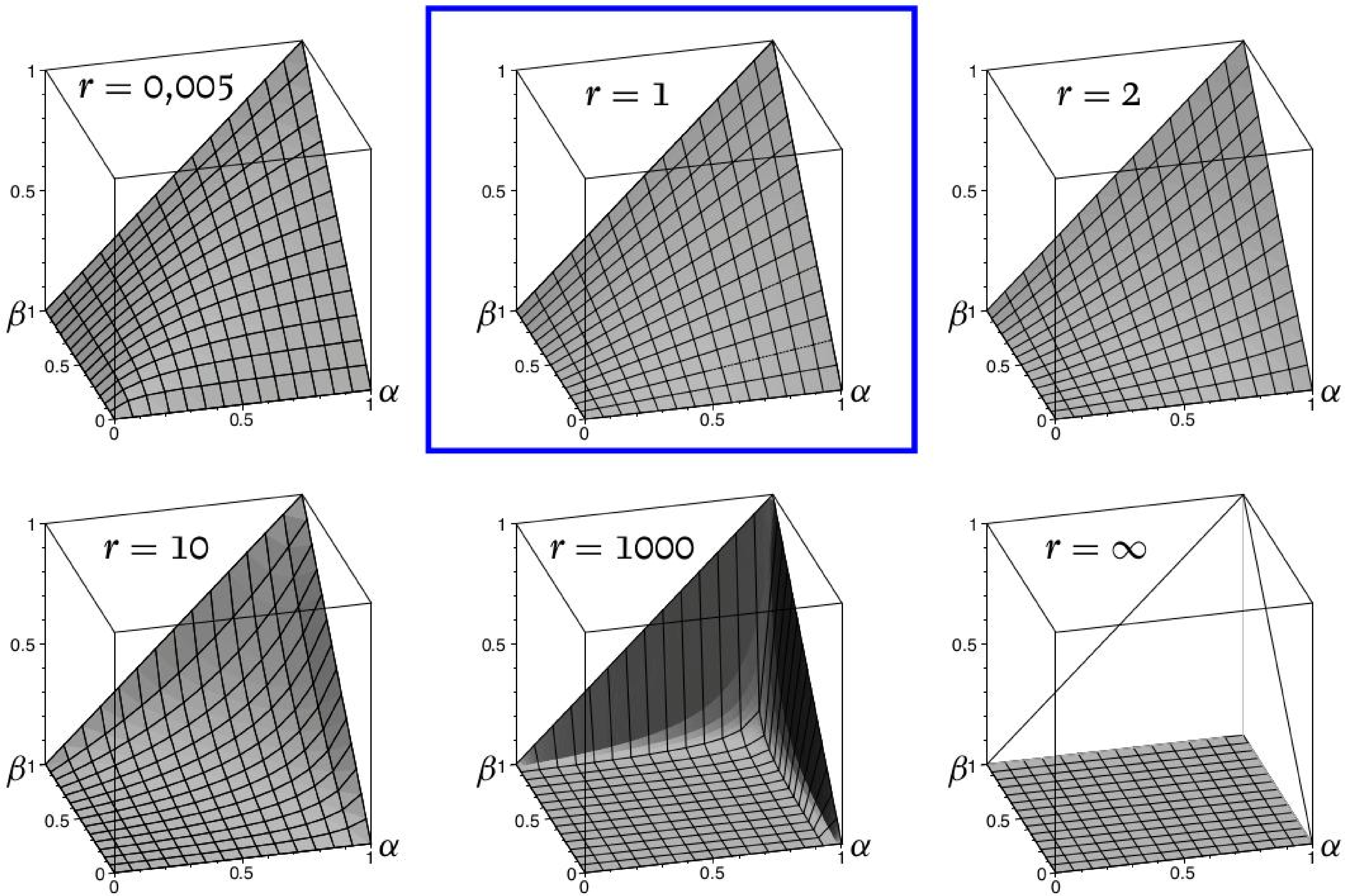

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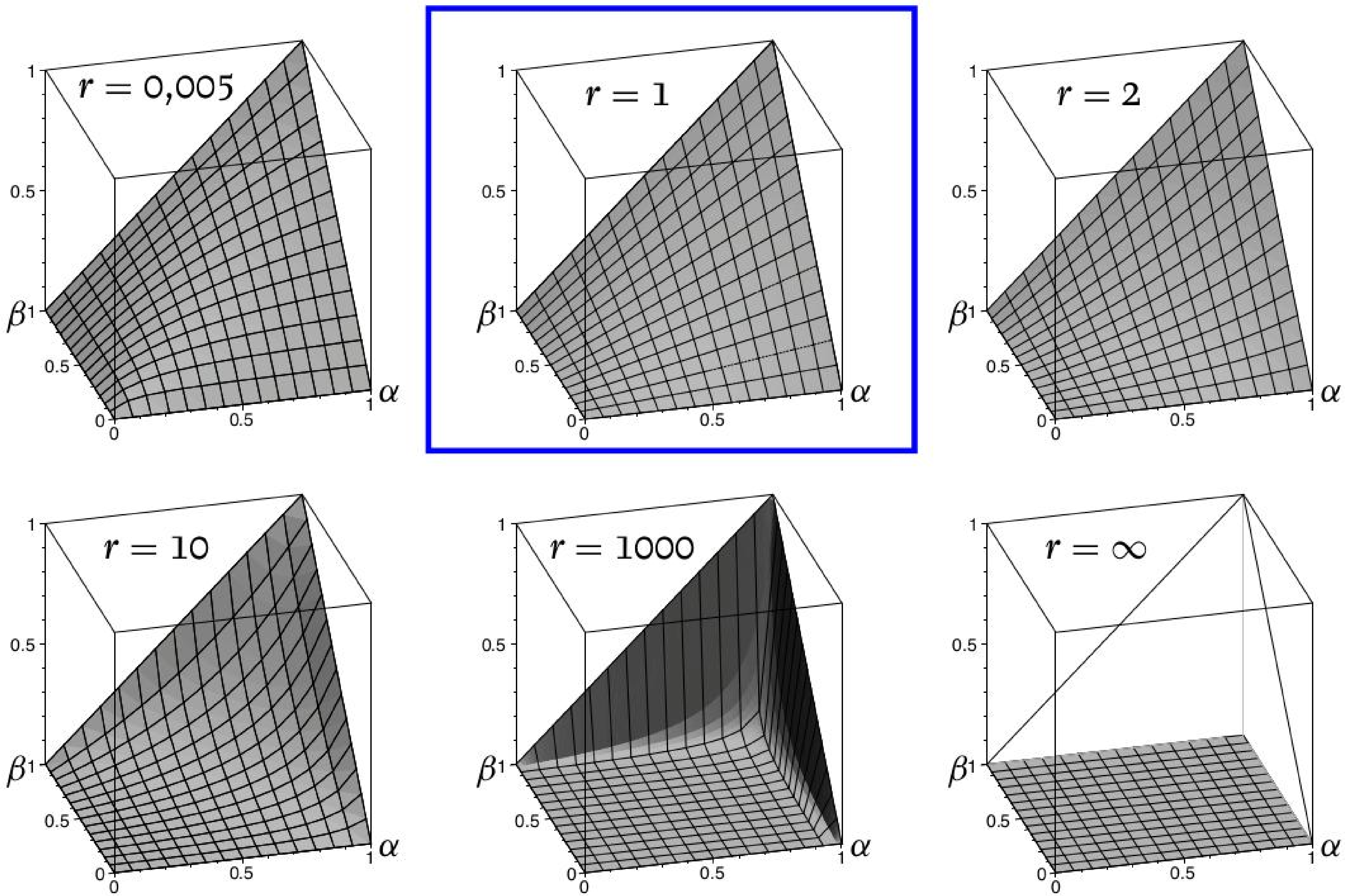

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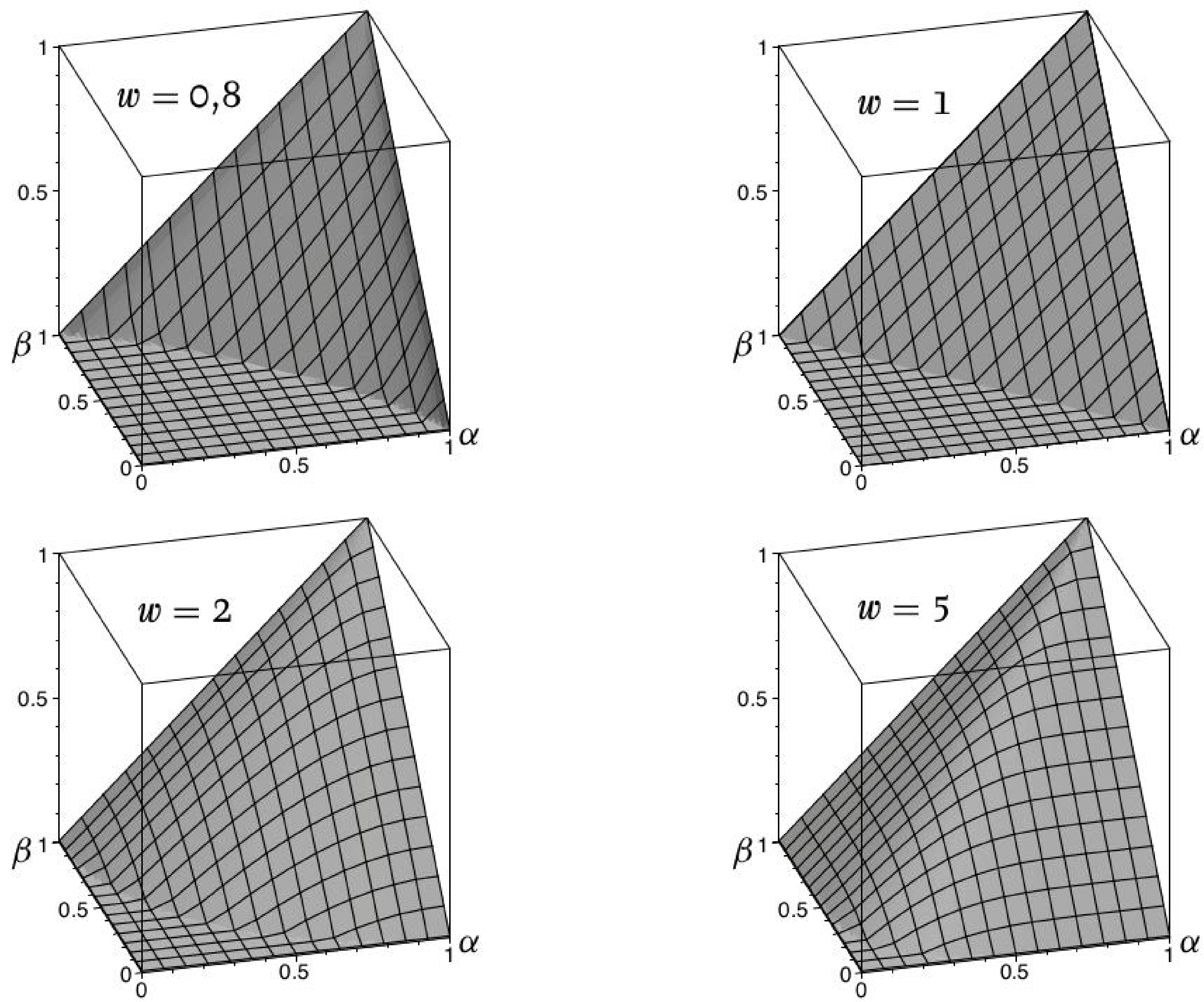
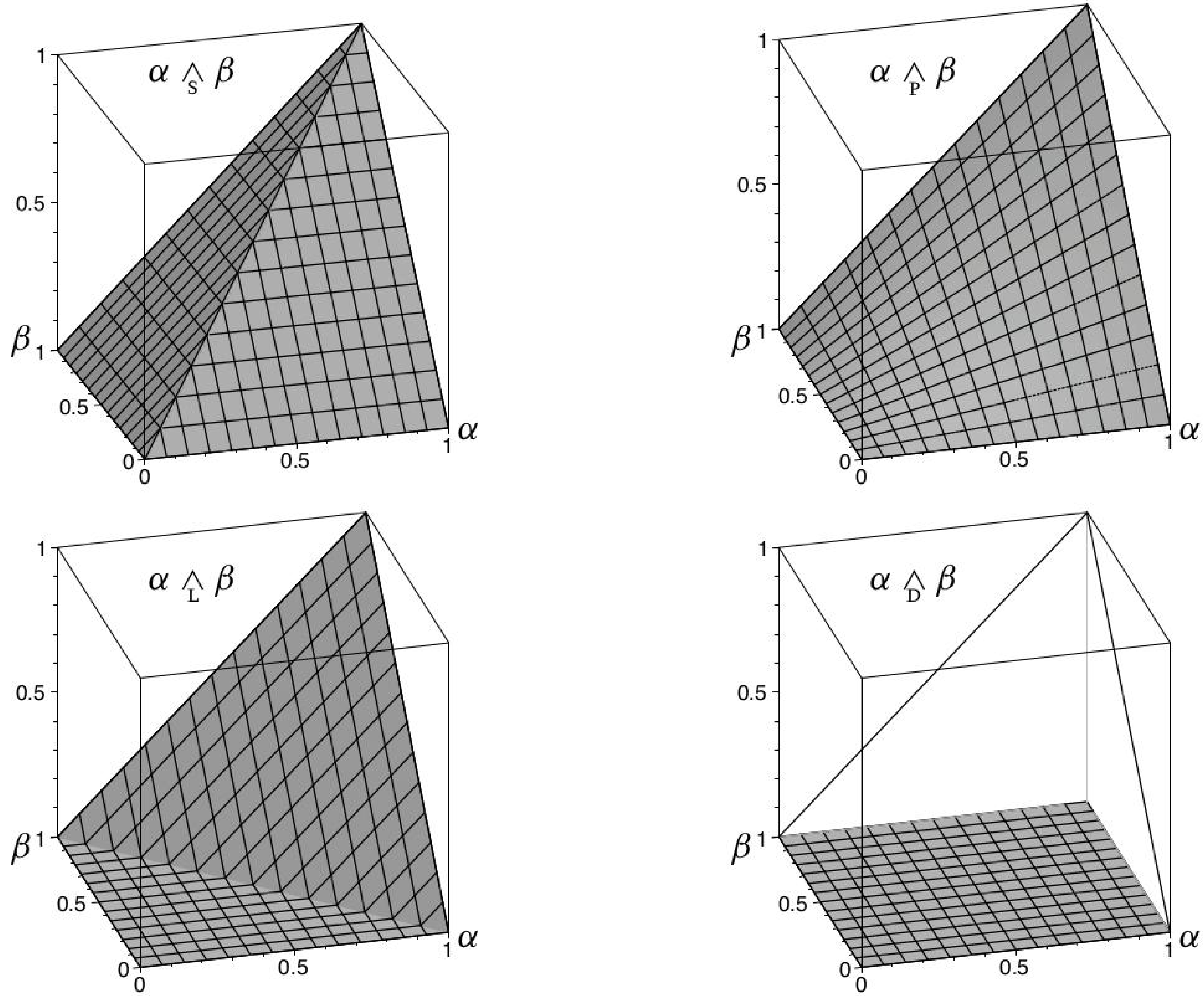


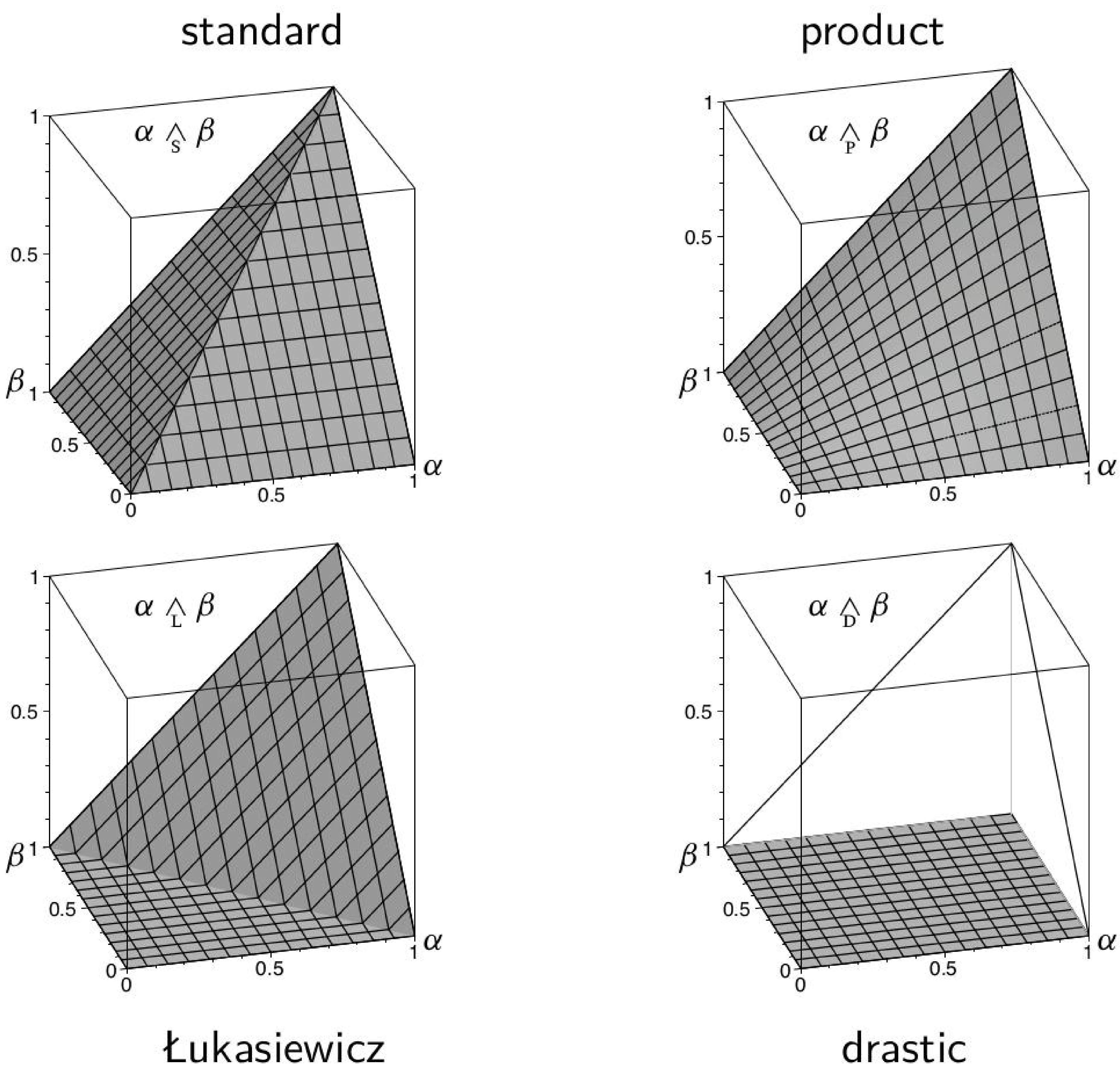
Basic fuzzy conjunctions

standard product



Łukasiewicz drastic

Basic fuzzy conjunctions



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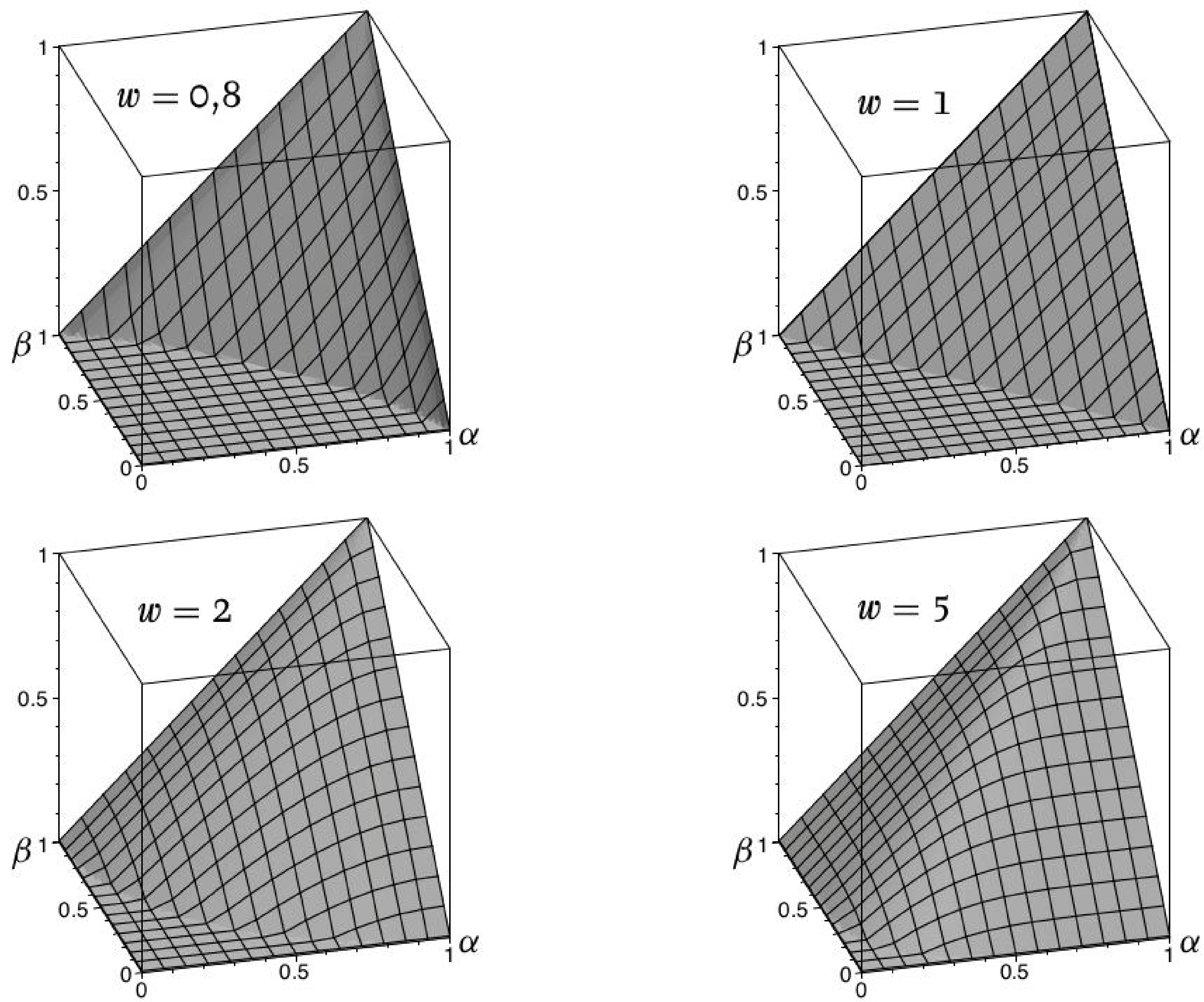

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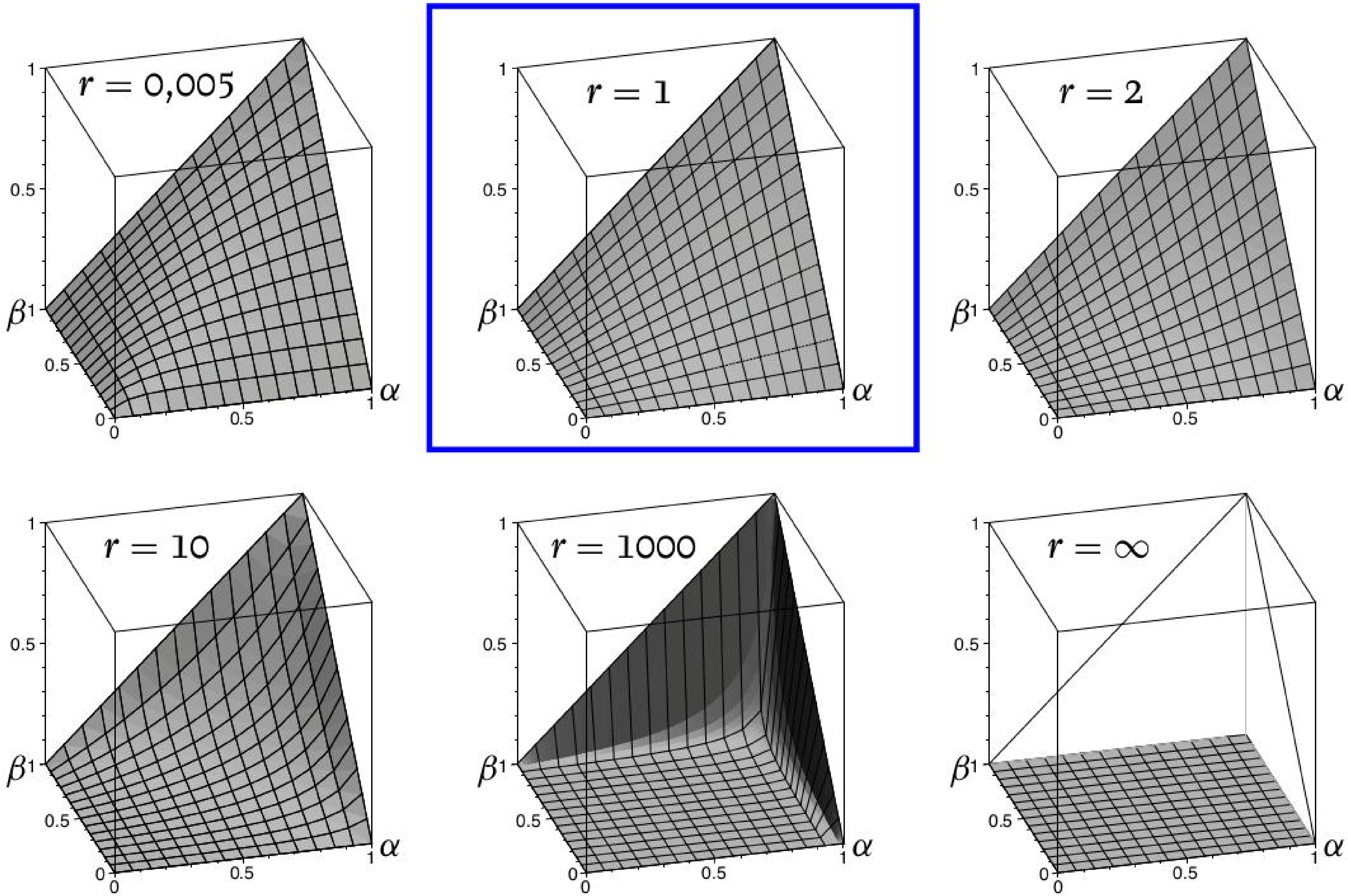

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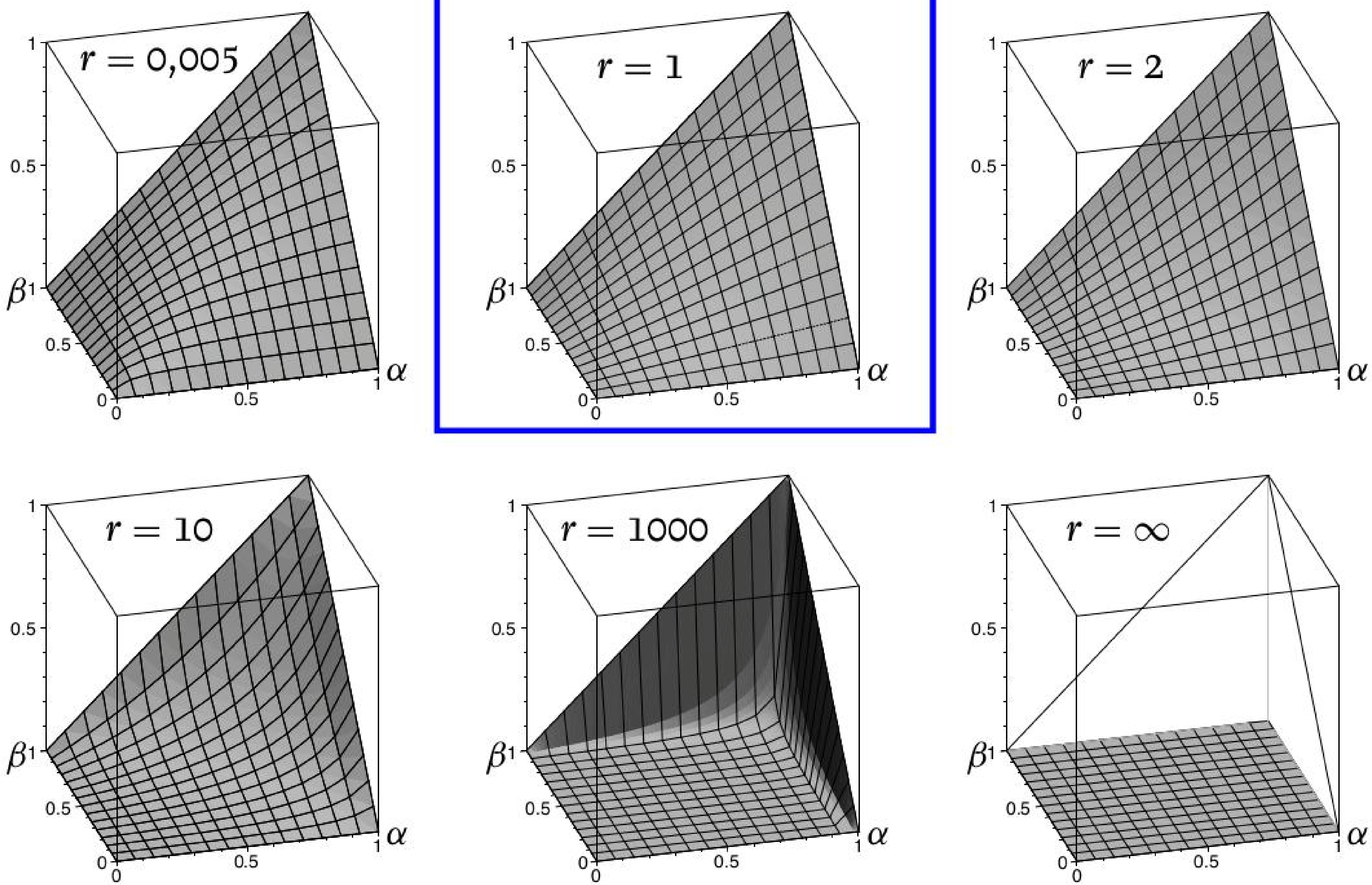

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Theorem: Standard conjunction is the only one which is **idempotent**, i.e.,
$\forall \alpha \in [0,1] : \alpha \underset{\cdot}{\wedge} \alpha = \alpha$

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Analogously for $\alpha > \beta$.

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Representation of fuzzy conjunctions (in general)

Theorem: Let $\underset{1}{\wedge}$ be a fuzzy conjunction and $i : [0,1] \to [0,1]$ be an increasing bijection. Then the operation $\underset{2}{\wedge} : [0,1]^2 \to [0,1]$ defined by

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is a fuzzy conjunction. If $\underset{1}{\wedge}$ is continuous, so is $\underset{2}{\wedge}$.

Proof:

- Commutativity (analogously for associativity):

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Classification of fuzzy conjunctions

Continuous fuzzy conjunction $\underset{\cdot}{\wedge}$ is

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Representation theorem for strict fuzzy conjunctions

Operation $\underset{\cdot}{\wedge} : [0,1]^2 \to [0,1]$ is a strict fuzzy conjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**multiplicative generator**) such that

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